ECE 20875

Python for Data Science

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(Adapted from material developed by Profs. Milind Kulkarni, Stanley Chan, Chris Brinton, David Inouye, Qiang Qiu)

estimation and sampling

why sample?

- Most analysis problems do not let you work with the whole **population**, e.g.,
  - *How many engines have a defect?* Cannot take apart every engine to find out
  - *What is the average height of people in Indiana?* Would be nearly impossible to measure every person in the state
  - *What is the difference in commute times between people in Indianapolis and people in Chicago?* Again, cannot ask everyone in both cities
- We are often left trying to learn facts about a population by only studying a subset of that population, i.e., a **sample**

how to sample?

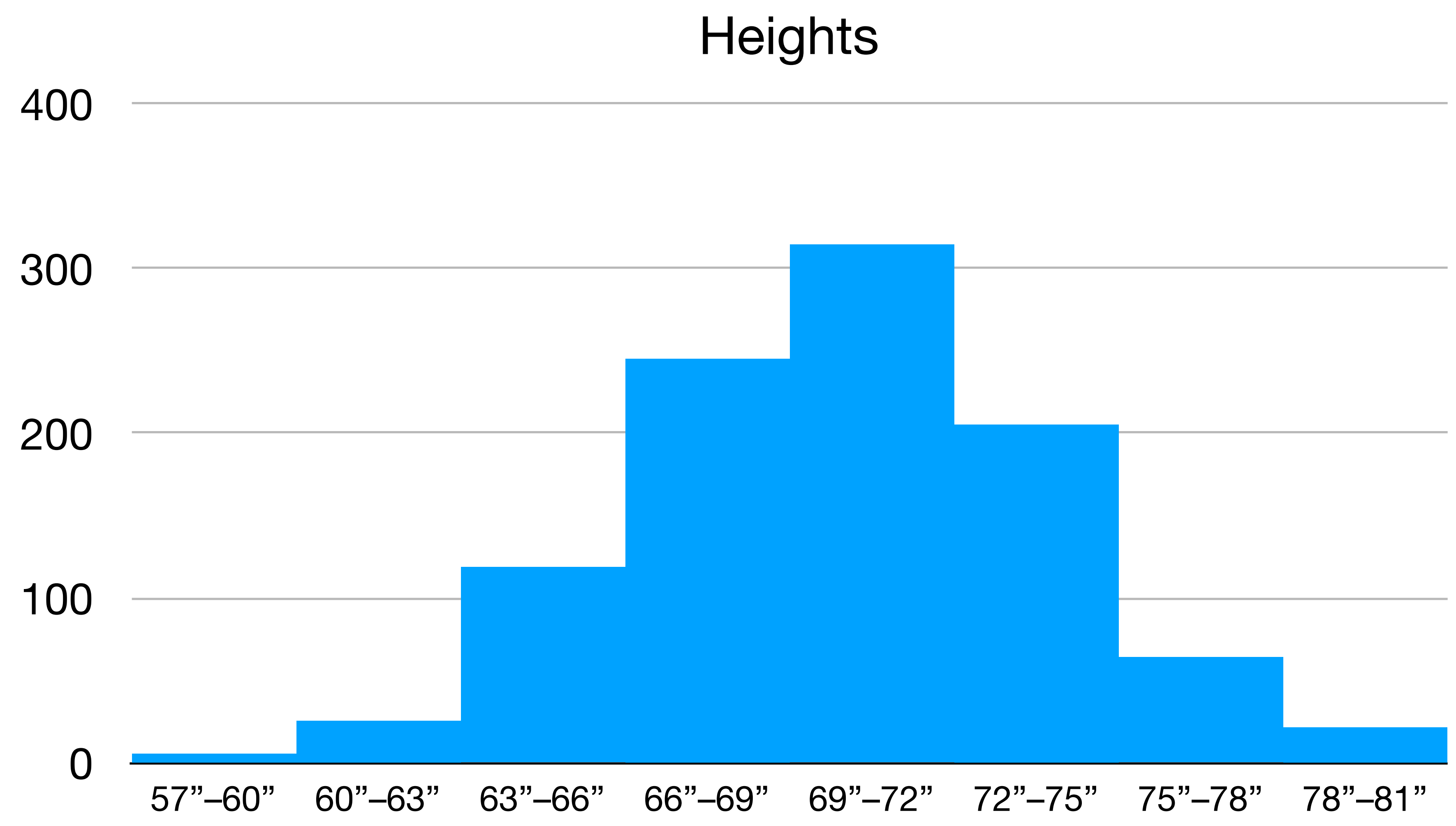
- Many strategies. Some common techniques:
  - **Simple Random Sampling** (SRS): Select S elements from a population P so that each element of P is equally likely to appear in S. Easiest to analyze, but can make it hard to represent rare samples (rare groups won't show up).
  - **Stratified Sampling**: Subdivide population P into subgroups P_1, P_2, etc. where each subgroup represents a distinct attribute (e.g., breaking a population up by cities). Do SRS within the subgroups, and combine the result. Ensures representation of each subgroup, but can be hard to set up.
  - **Cluster Sampling**: Group population into random clusters (not specific subgroups like in stratified sampling). Select clusters at random, add all elements from selected clusters to sample. Easier to conduct than SRS, but adds more variability.
- We will focus mainly on SRS in this course

statistic vs parameter

- We differentiate between attributes of the population and the sample
- Numbers which summarize a population are called **parameters**
  - Population mean (μ), variance (σ^2), median, etc.
- Numbers which summarize a sample are called **statistics**
  - Sample mean ($\bar{x}$), variance (s^2), median, etc.
  - The statistics are not guaranteed to be close to the parameters (why?)
- **Estimation** is the problem of making educated guesses for parameters given sample data
  - Key question: How close is our estimate to the true parameter?

sampling

- Let’s consider a population of 1000 people whose heights we have measured



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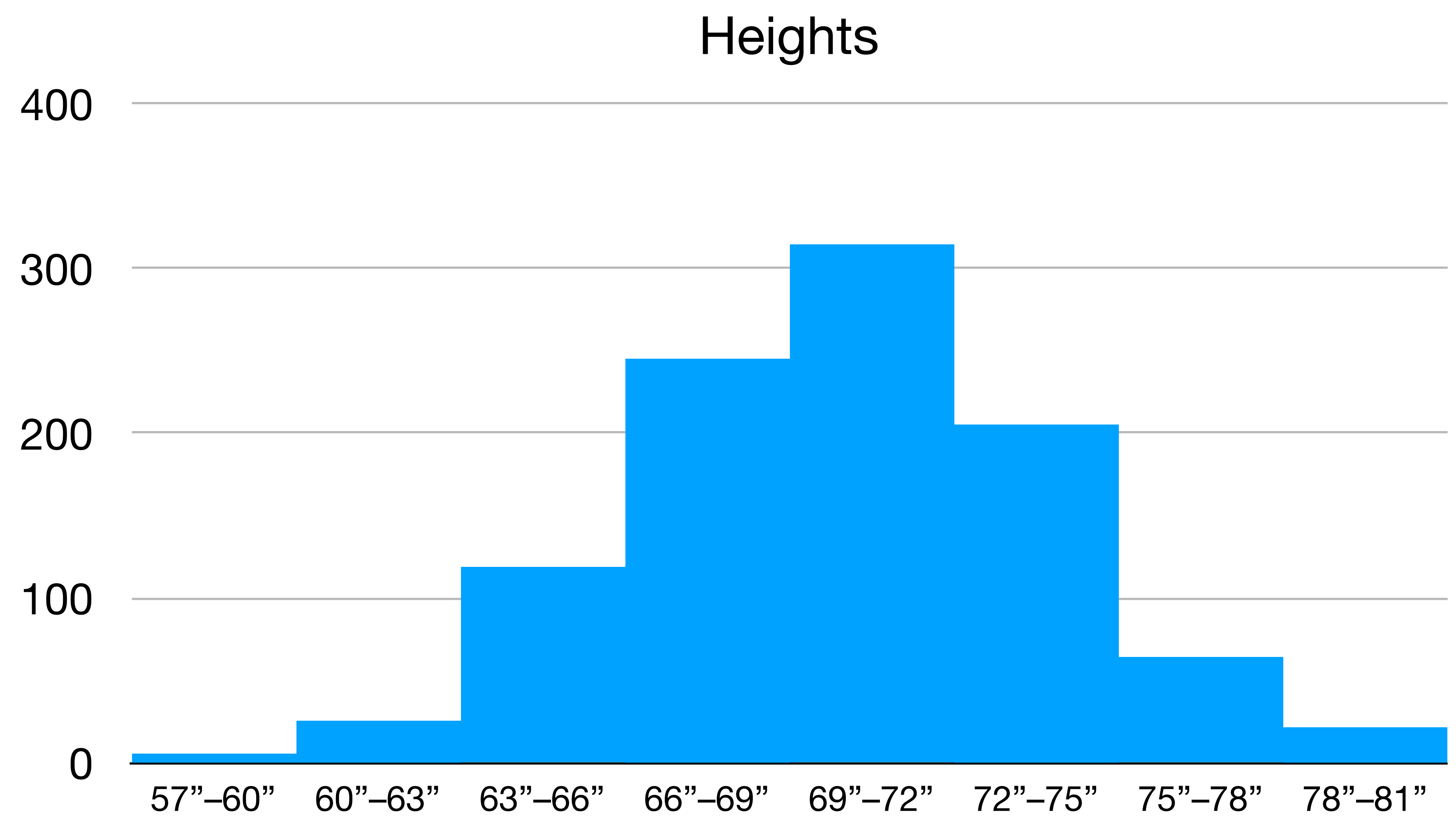
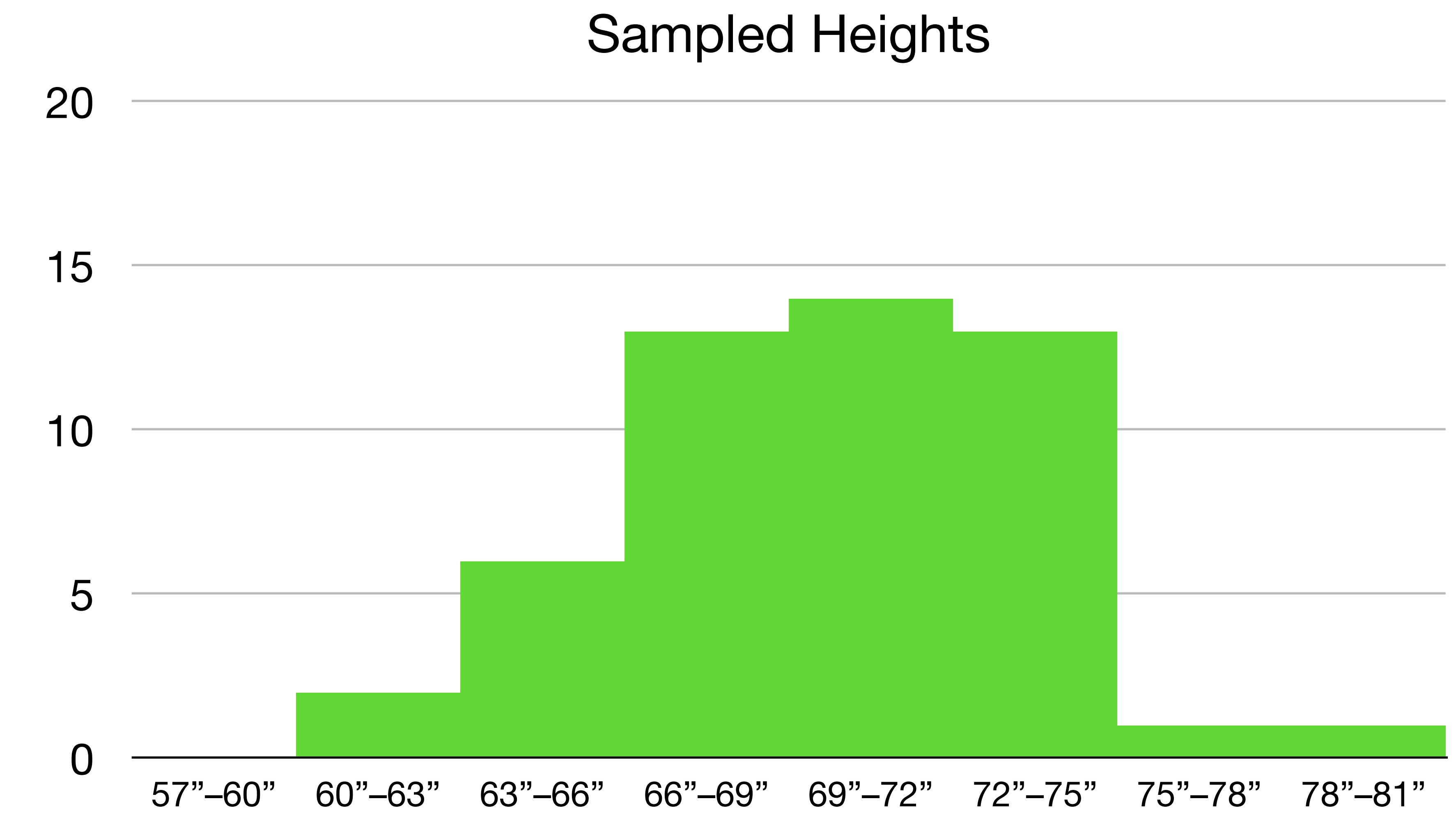
- What if we sample $n = 50$ of them at random?
  - Don't get exactly the same distribution





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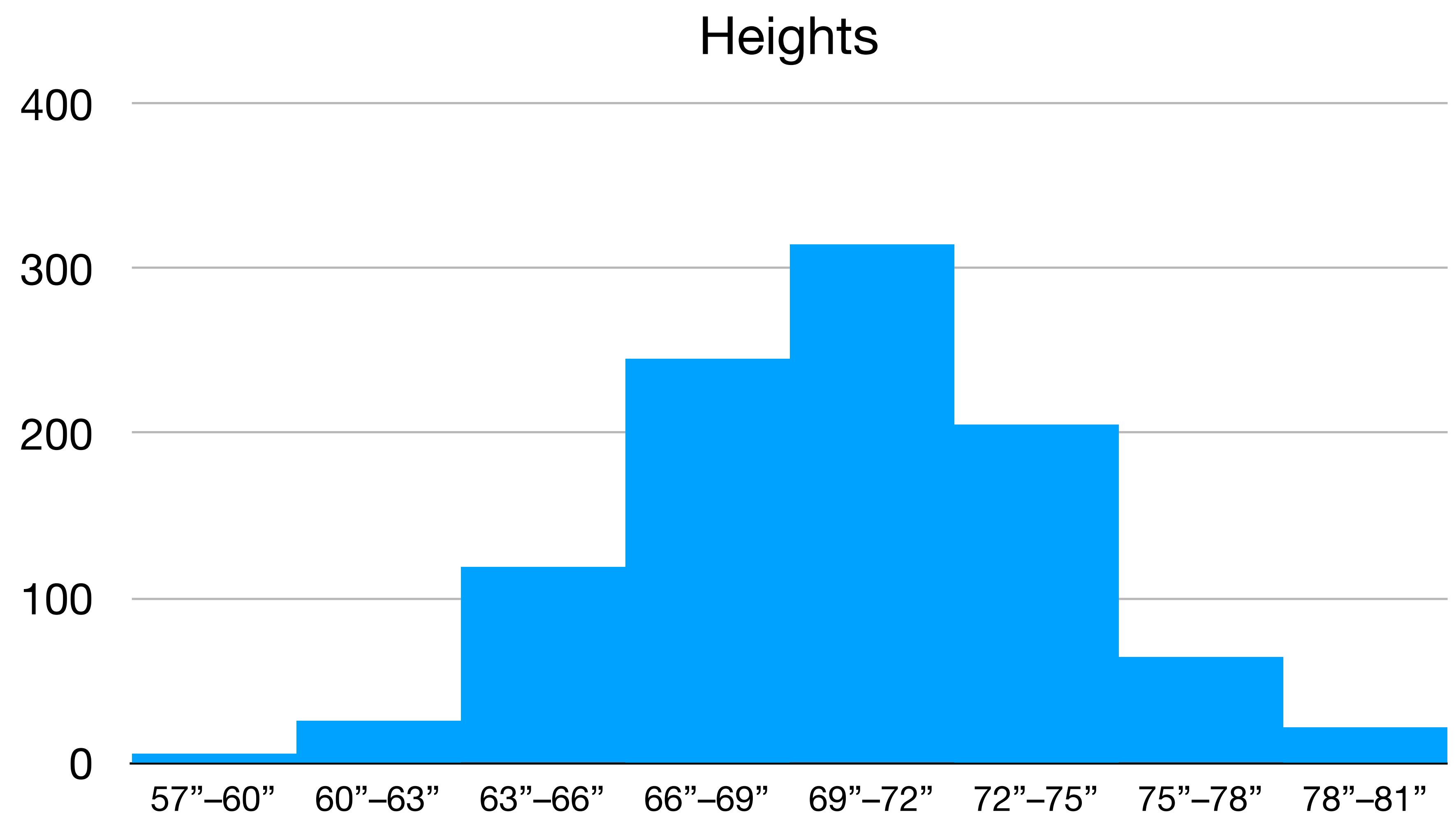

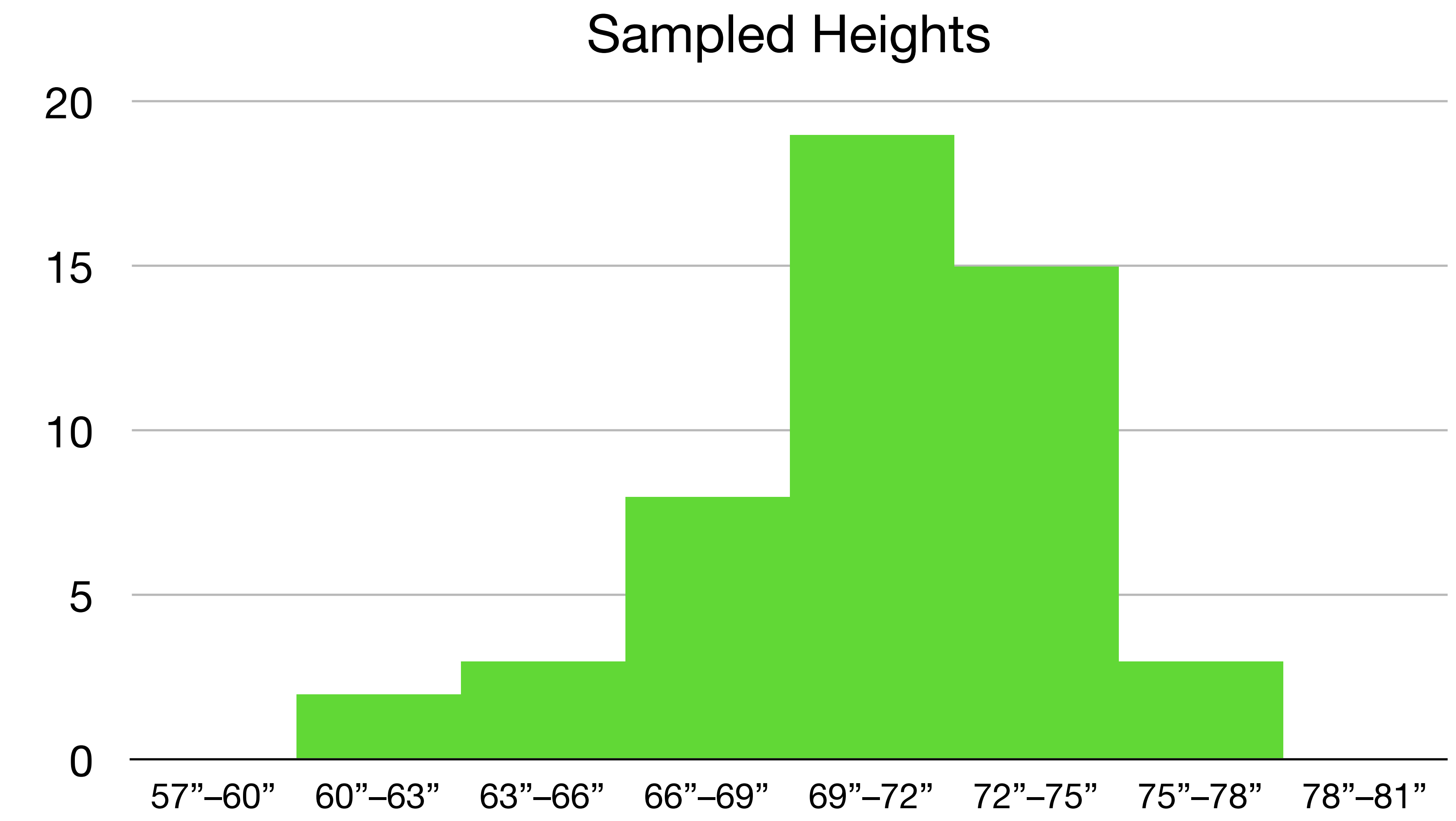

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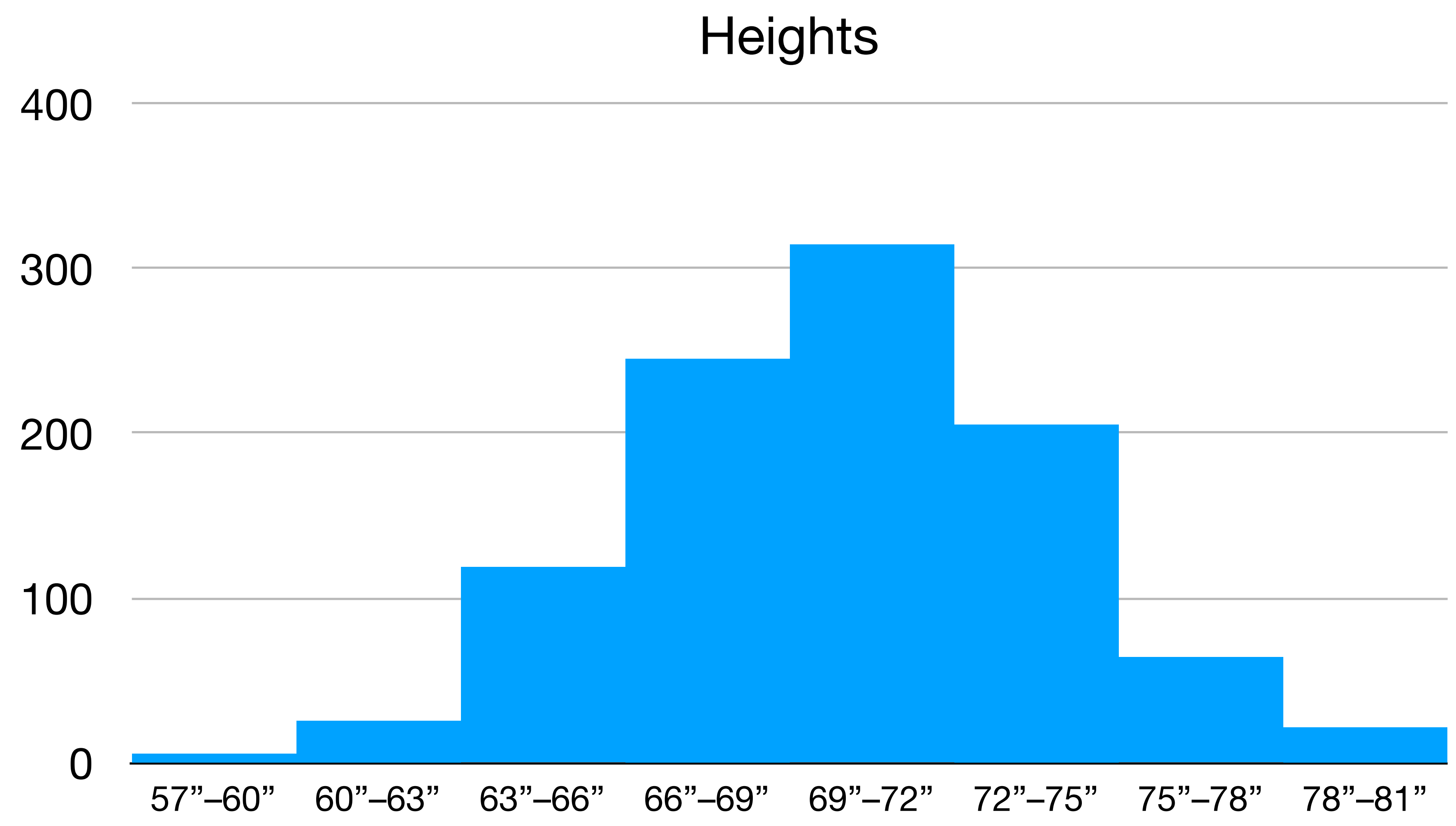
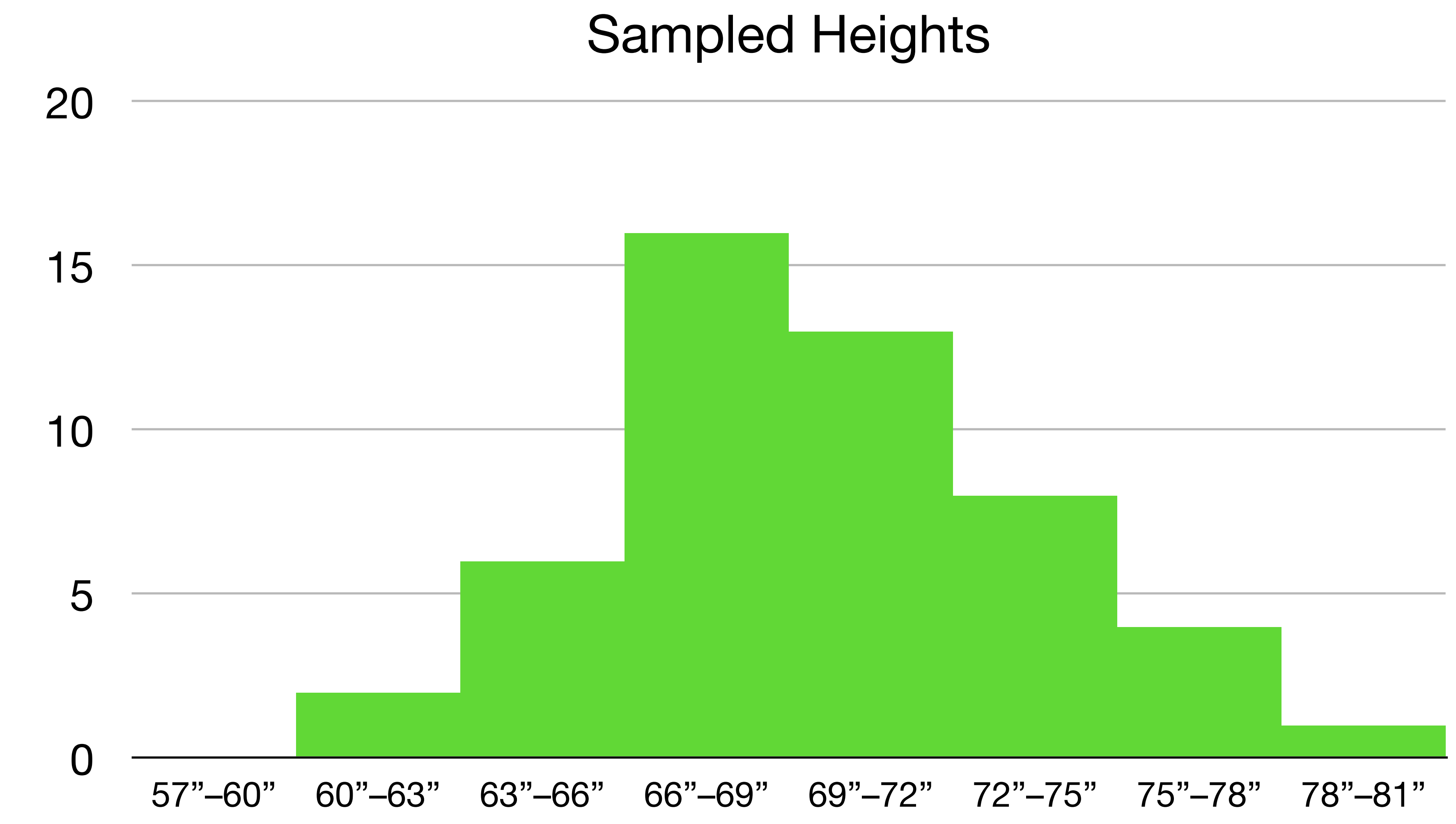
- And again?





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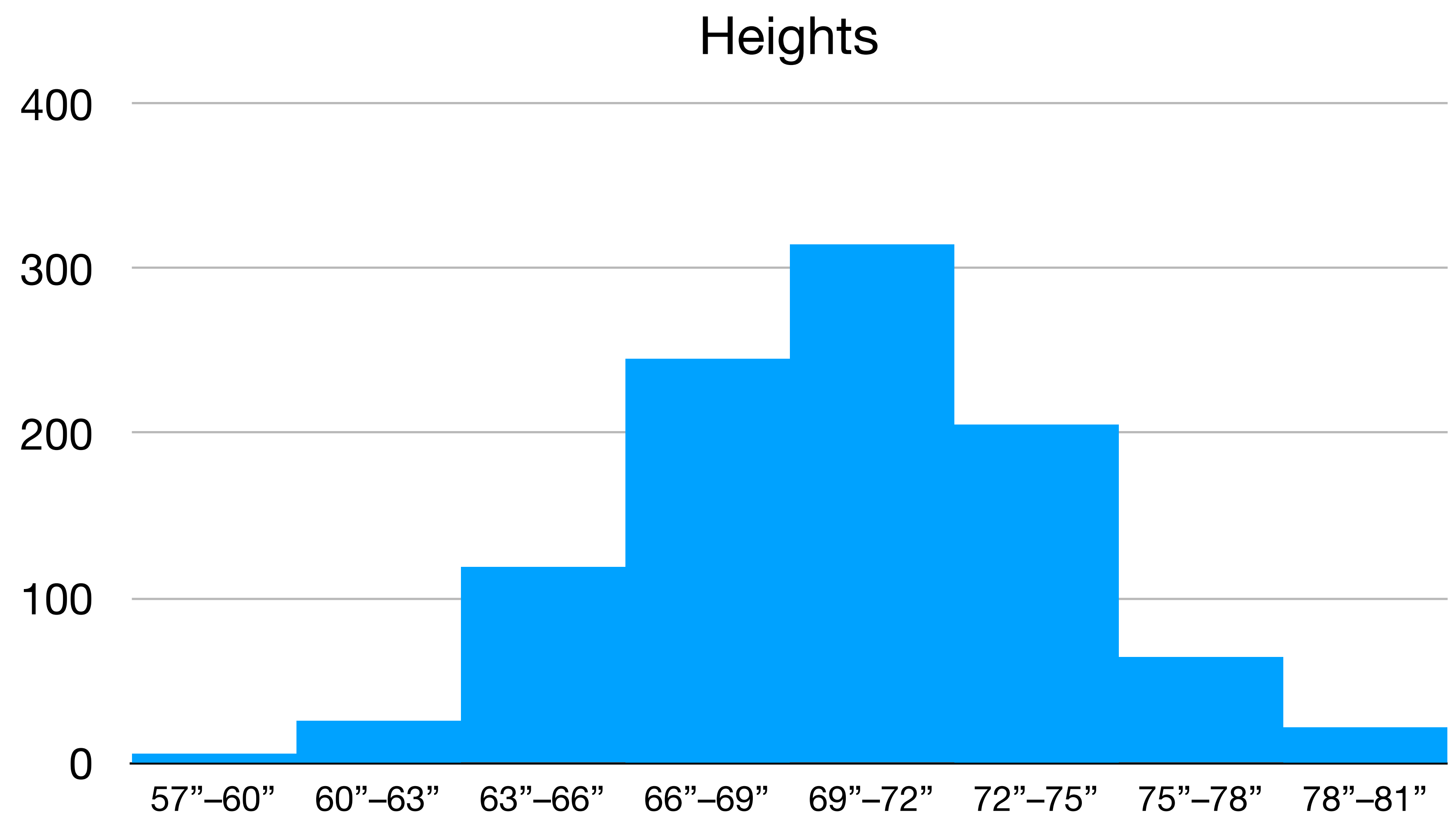

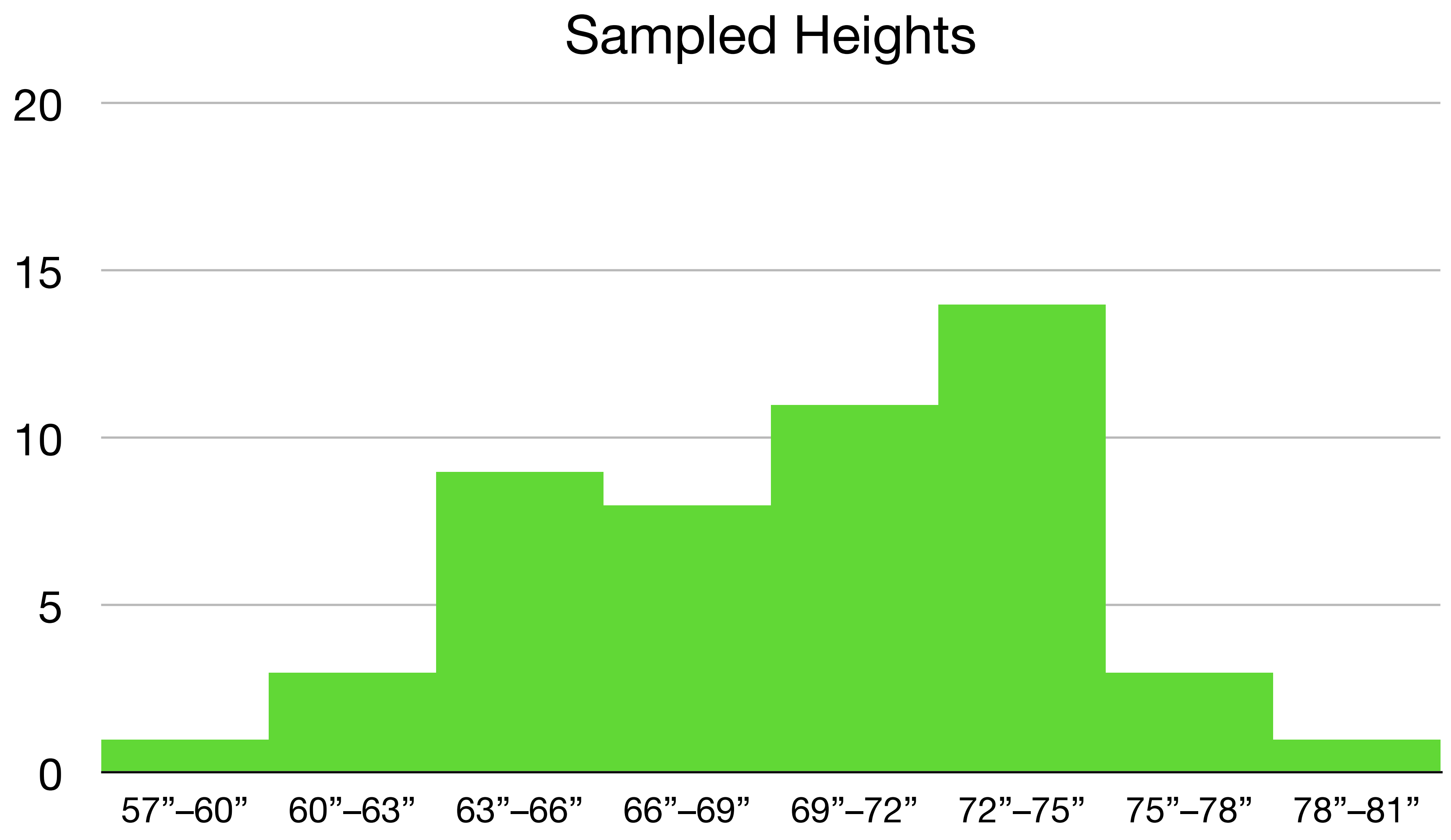

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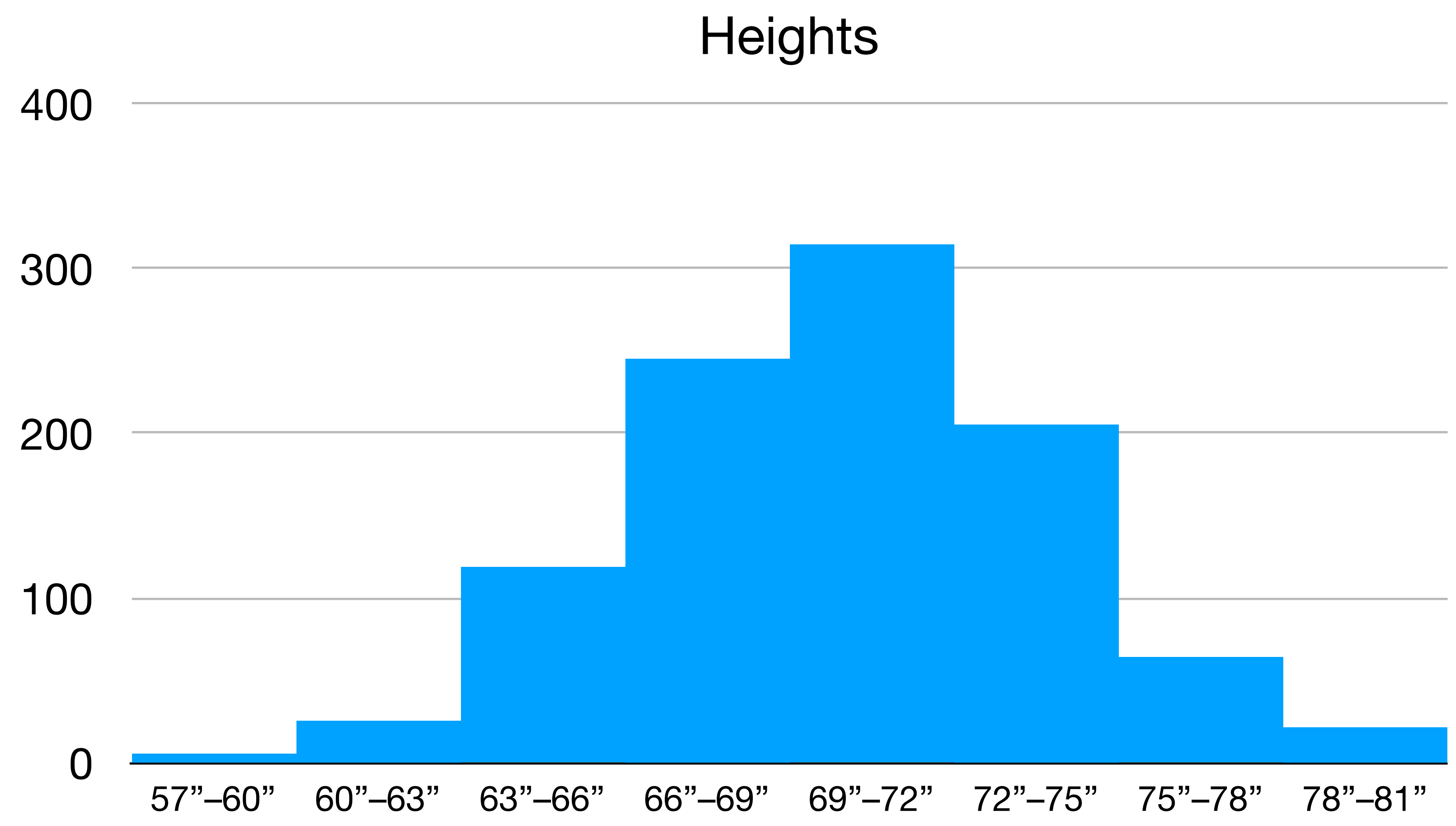
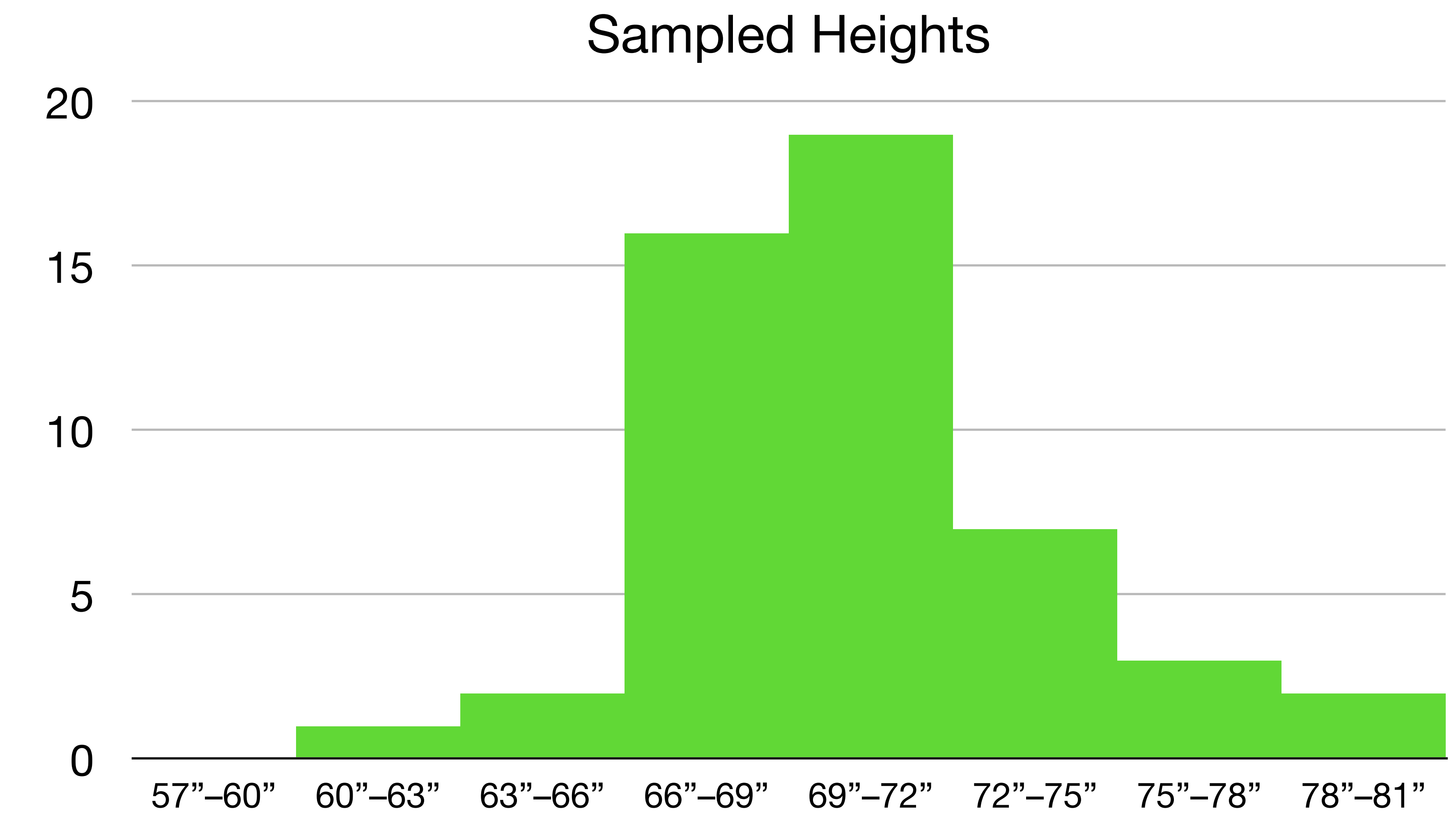




roadmap for estimating mean

- We want to estimate the *population mean* μ

- Let's estimate this by the *sample mean* $\bar{x}$ of $n = 50$ samples

- Key question: How close is $\bar{x}$ to μ?

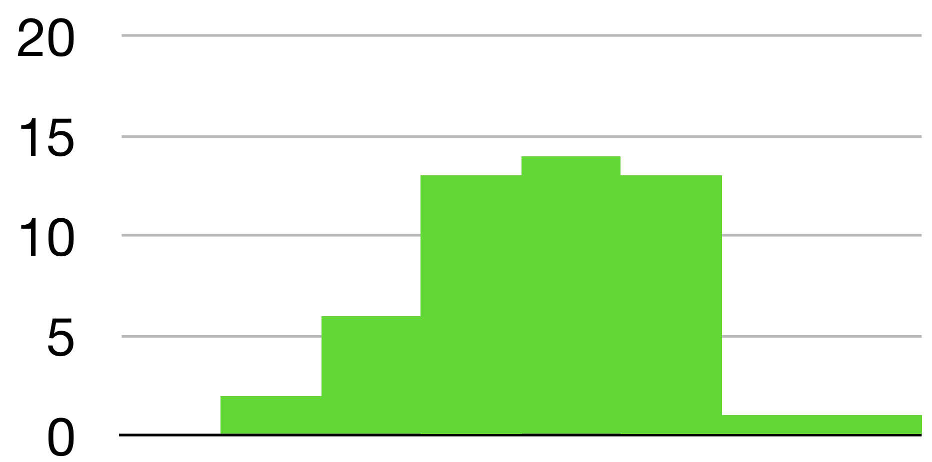
  - First, let's consider a *hypothetical scenario*: ***What if*** we could repeat this experiment as many times as we want and we knew μ?

  - Second, we will see that we can use *theory* to reason about this hypothetical (but unrealistic) scenario (leading to the **central limit theorem**)

  - Third, we will use this theory to help answer the above question (leading to hypothesis testing and confidence intervals)

estimate the mean

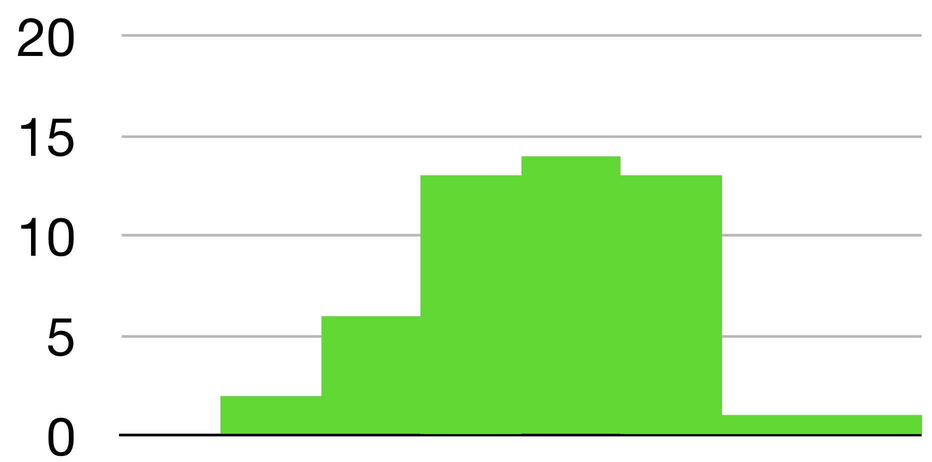
- What if we want to estimate the mean (μ) of a population?



$\bar{x} = 69.42$

estimate the mean

- What if we want to *estimate* the mean (μ) of a population?
- Can sample, and repeat the experiment



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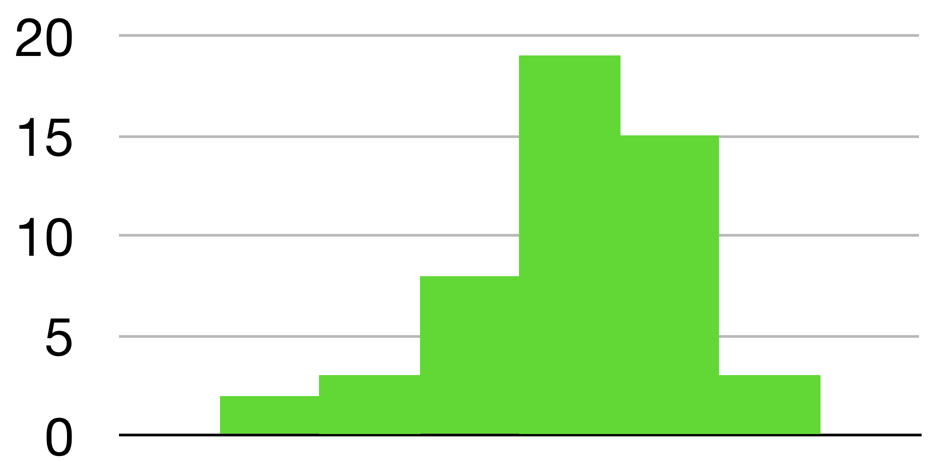
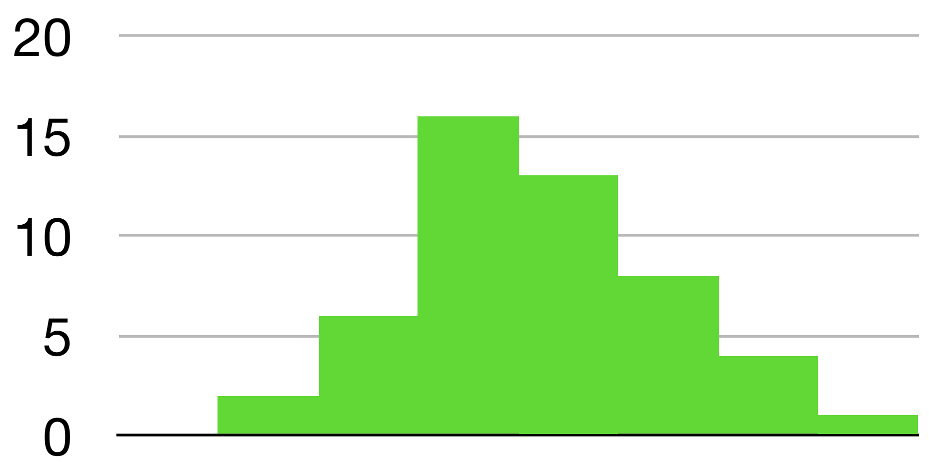


$\bar{x} = 70.02$



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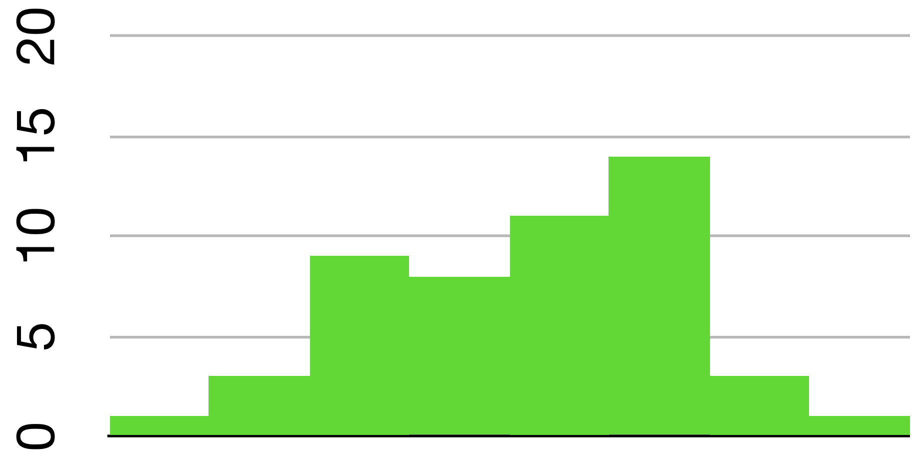

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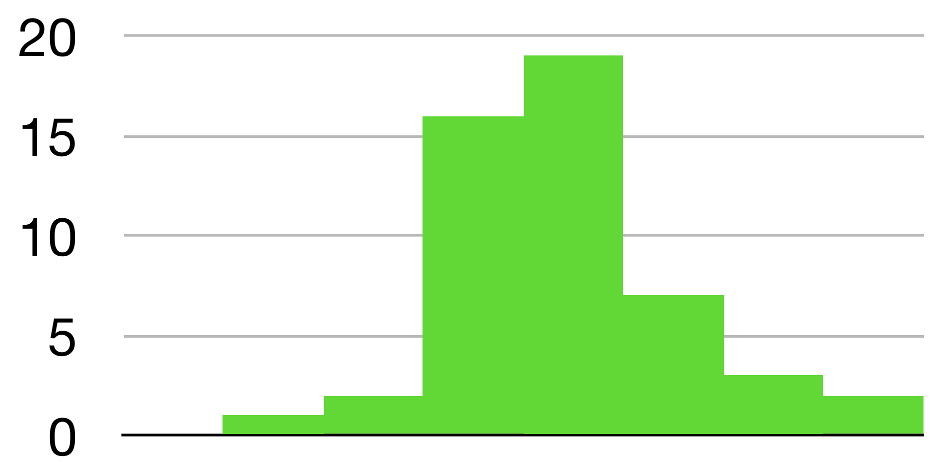

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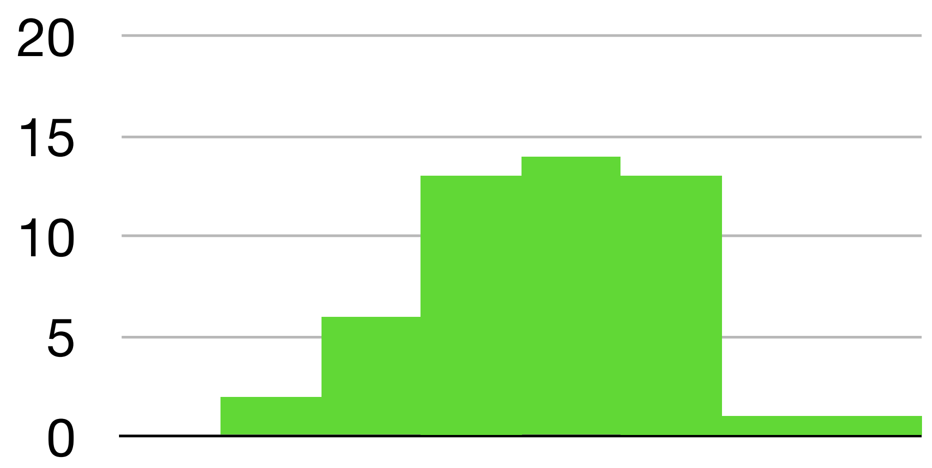
estimate the mean

- What if we want to *estimate* the mean (μ) of a population?
- Can sample, and repeat the experiment



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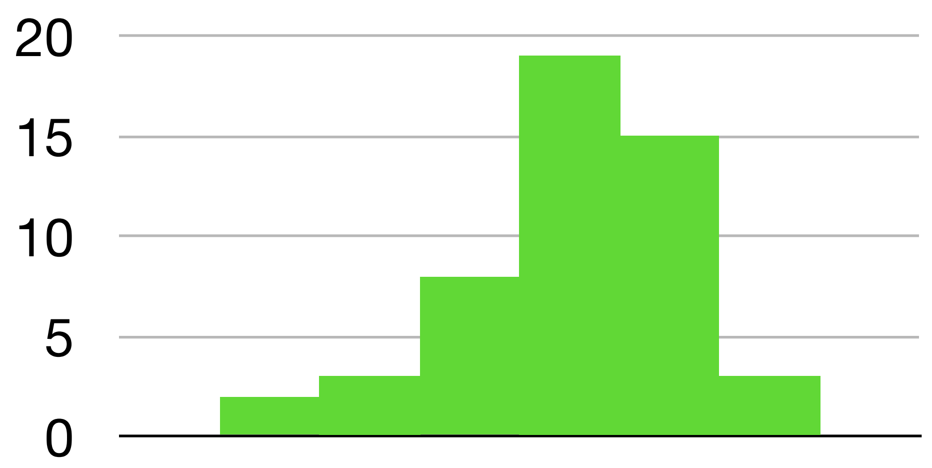

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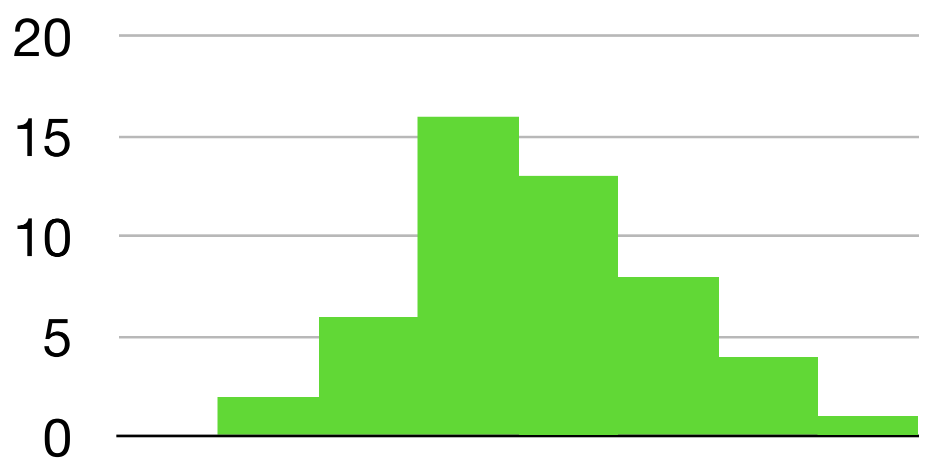

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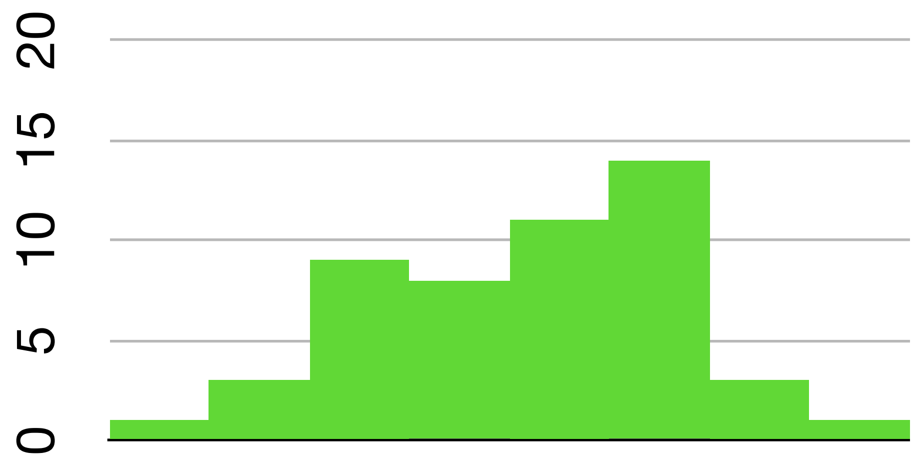

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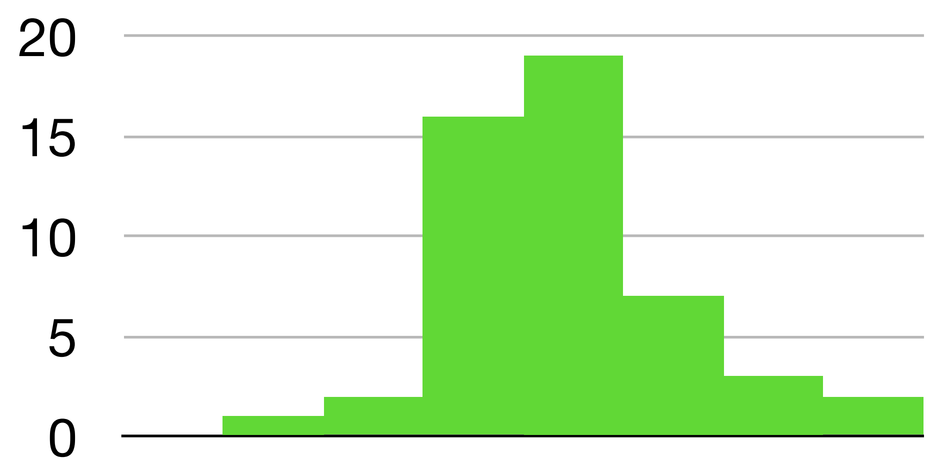

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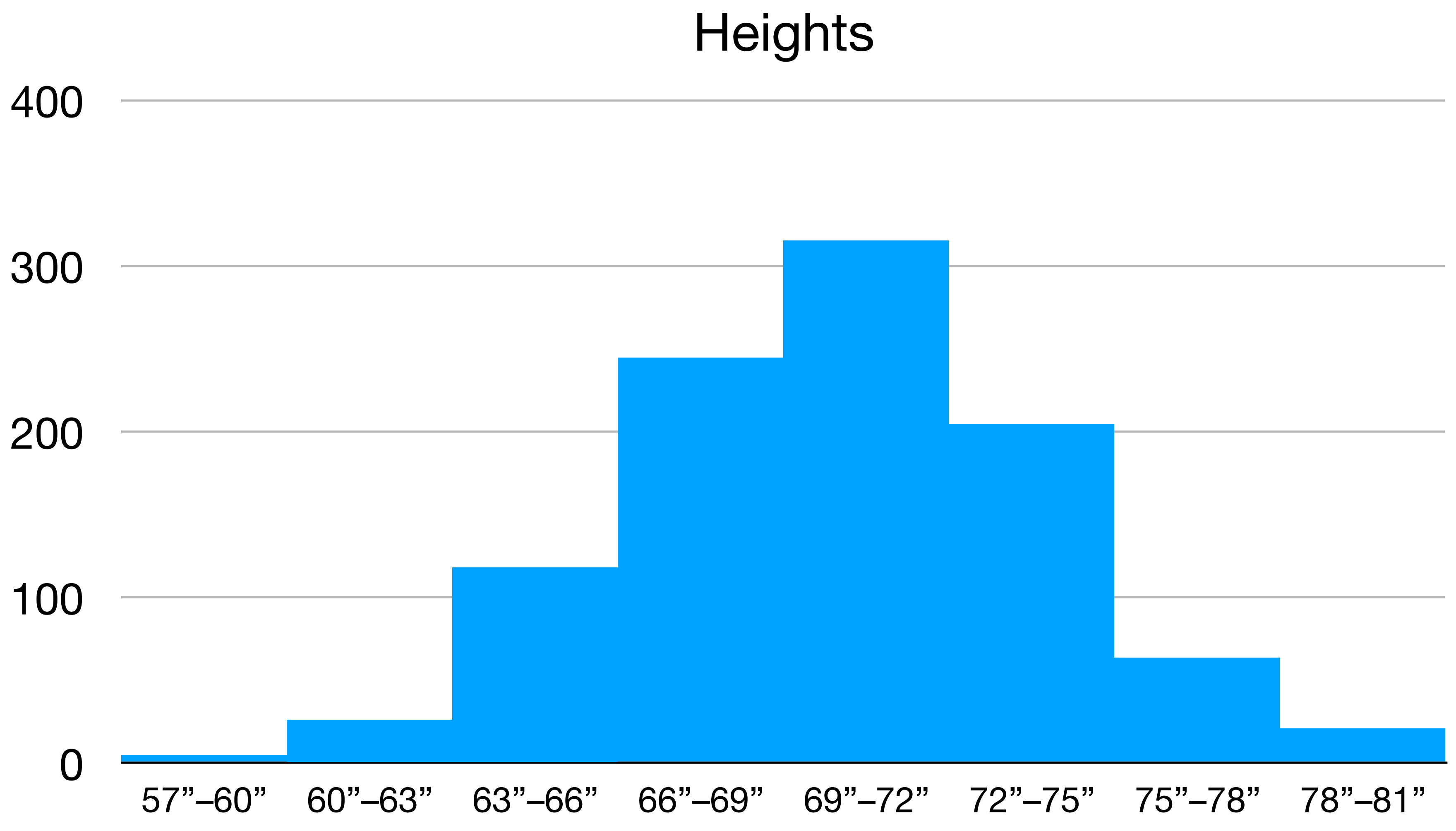
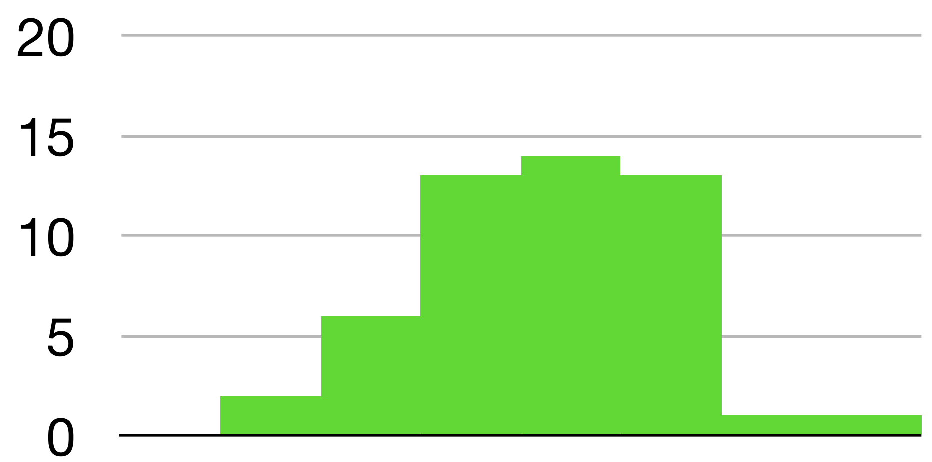


Population mean $\mu = 69.436$

estimate the mean

- What if we want to *estimate* the mean (μ) of a population?
- Can sample, and repeat the experiment



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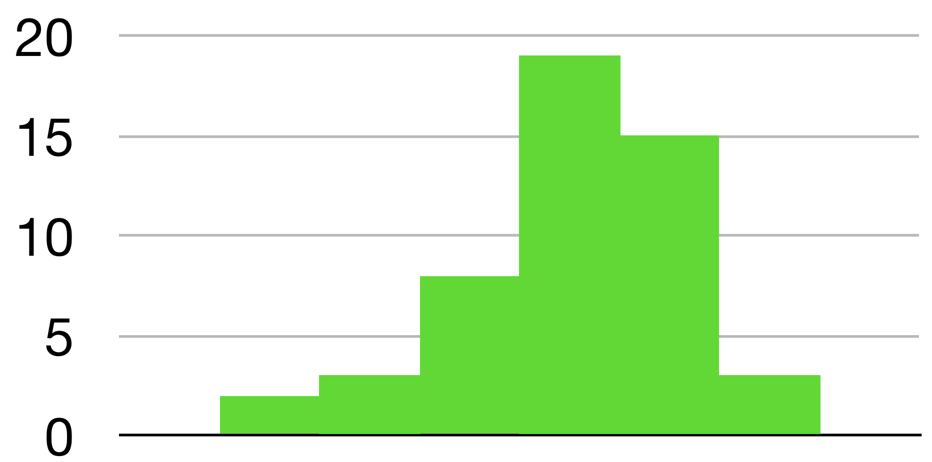

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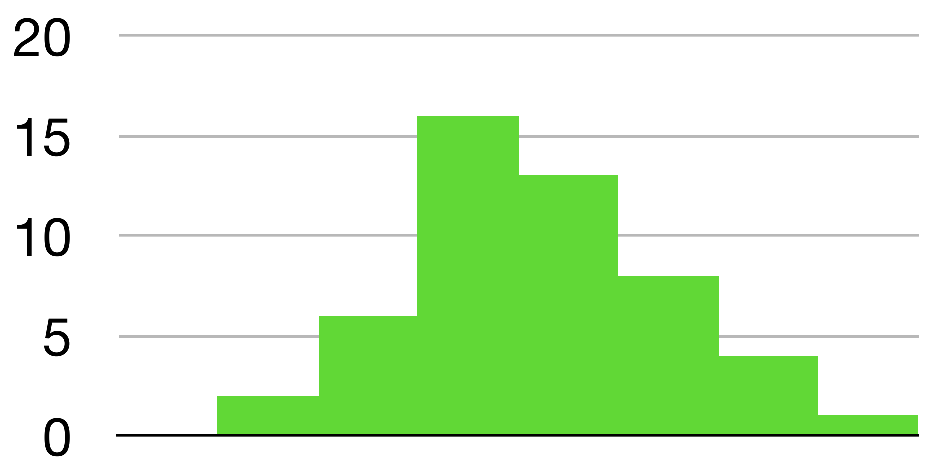

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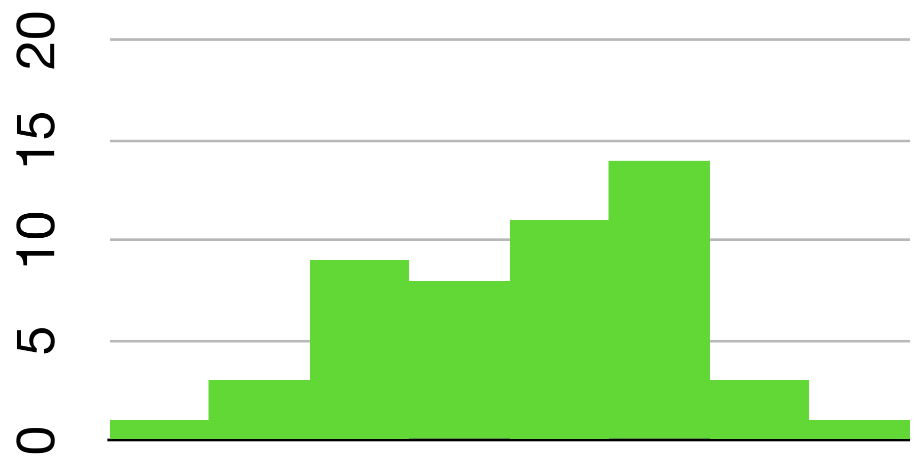

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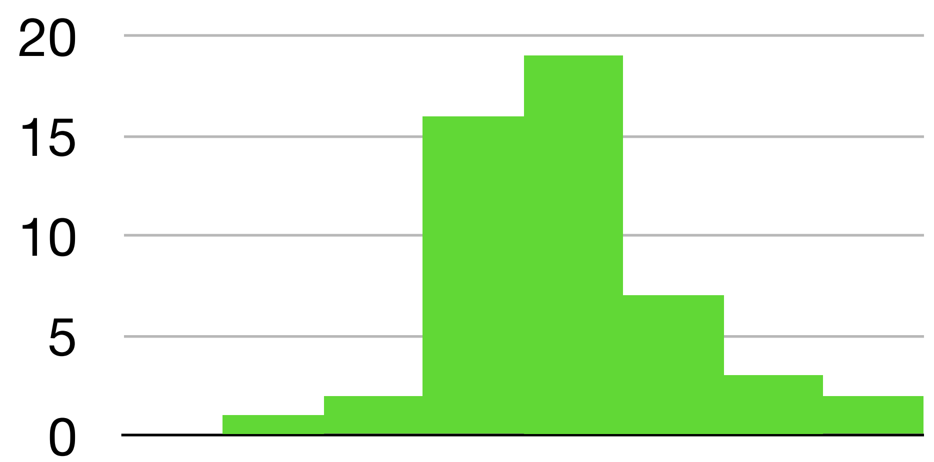

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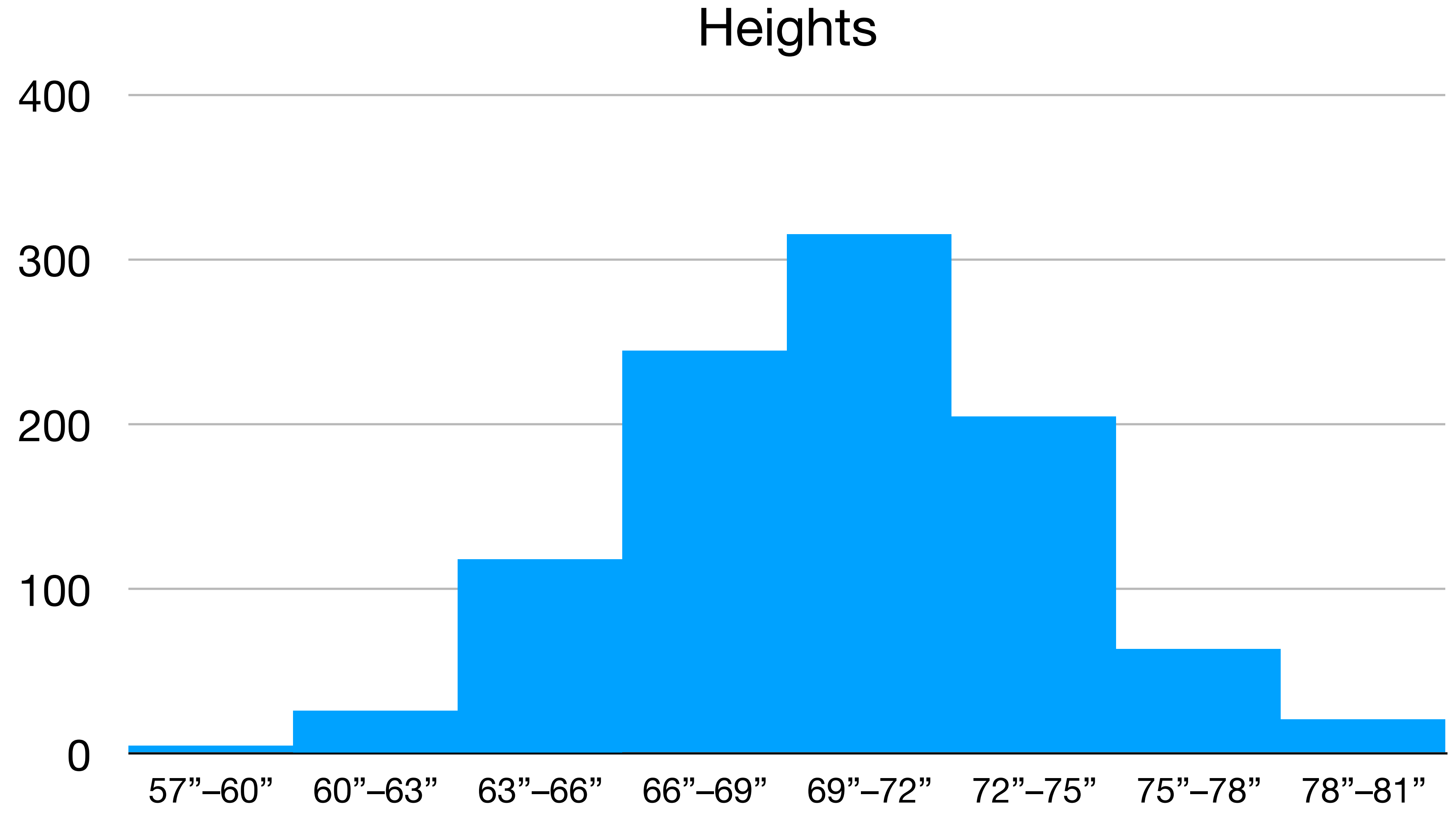
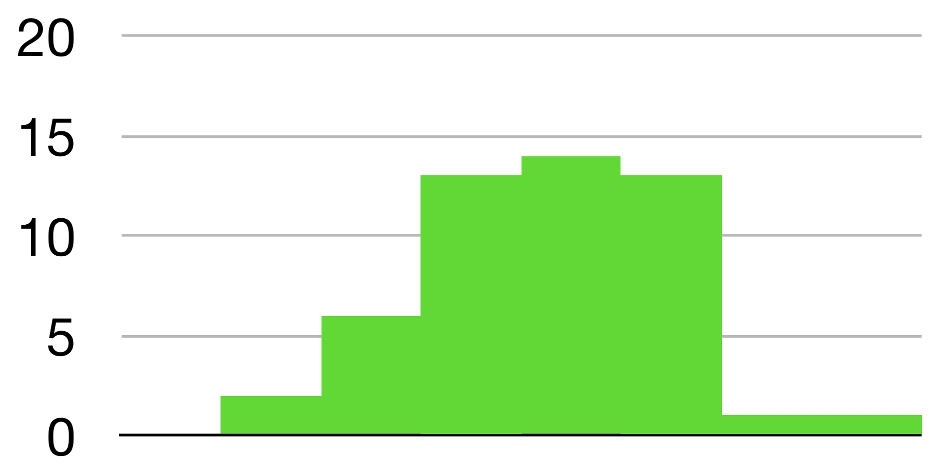


- Estimate μ of population using the sample $\bar{x}$’s based on each experiment
- How good is this estimate?
  - Use the mean squared error (MSE)

how good is our estimate?

- What if we want to *estimate* the mean (μ) of a population?
- Can sample, and repeat the experiment



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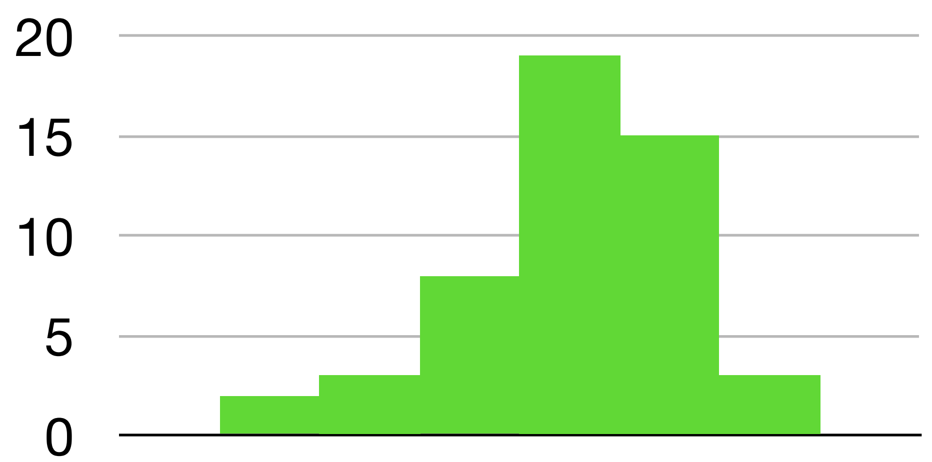

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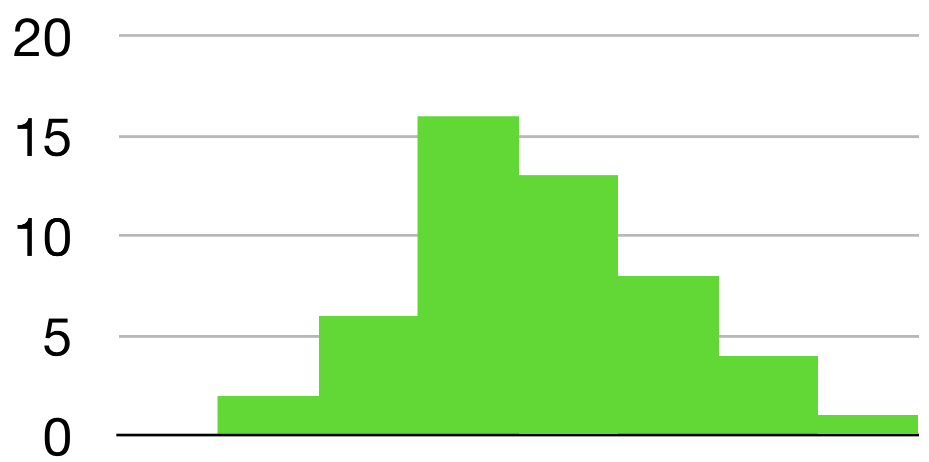

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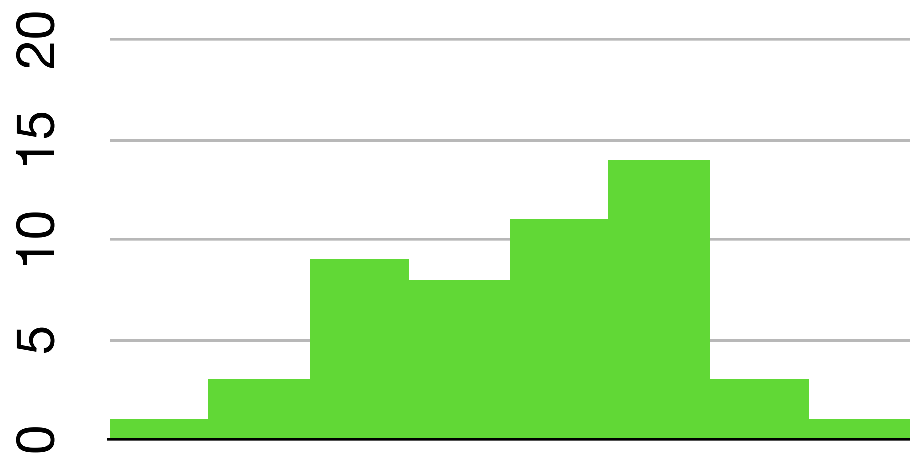

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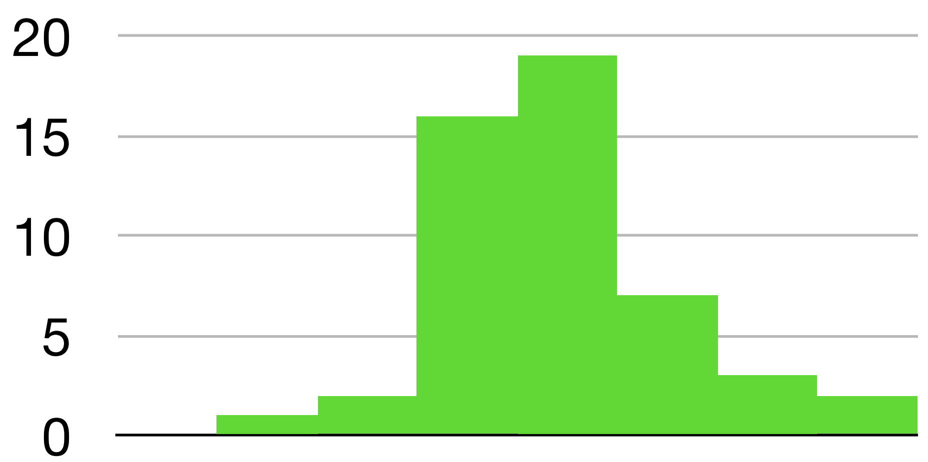

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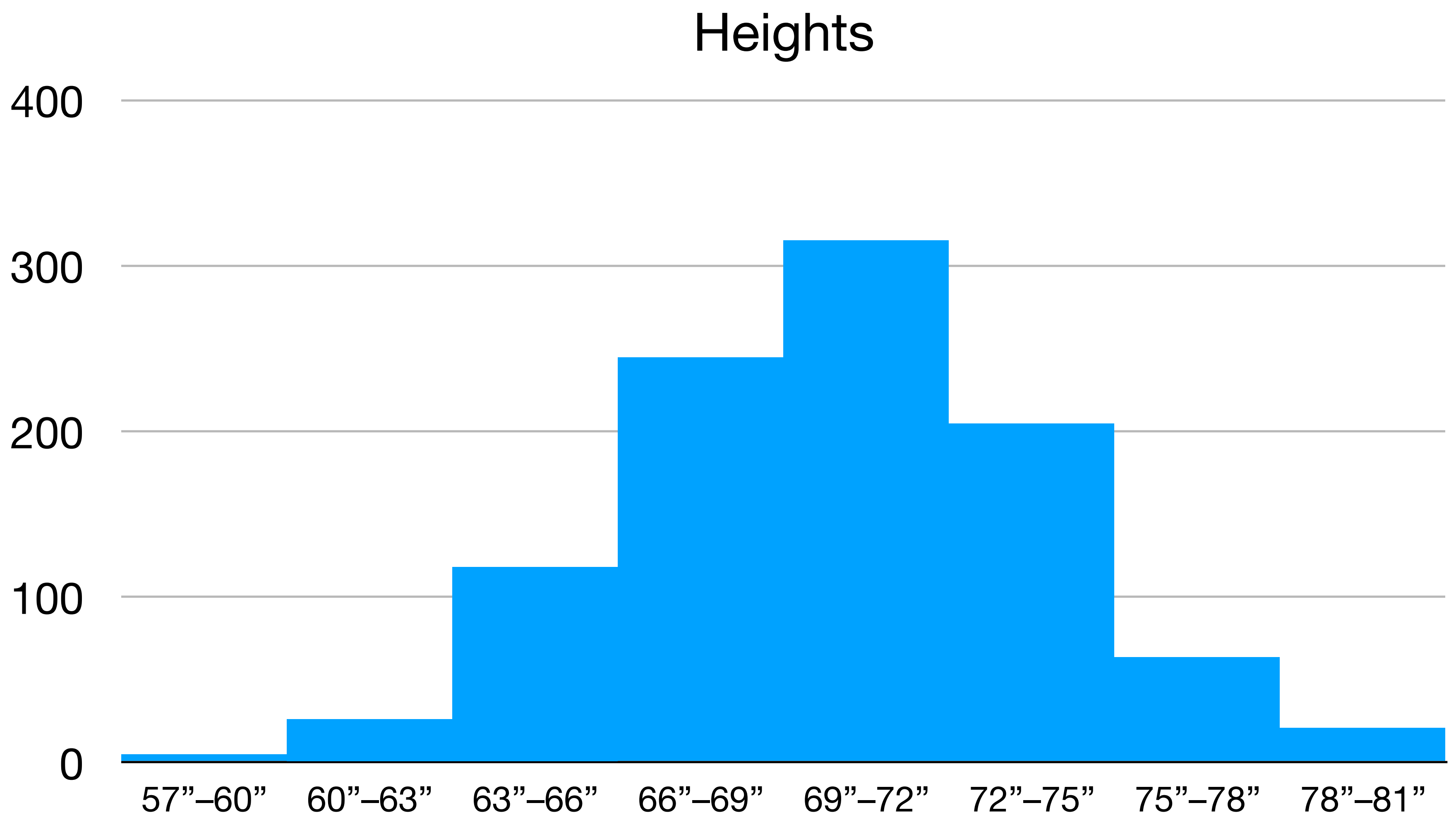


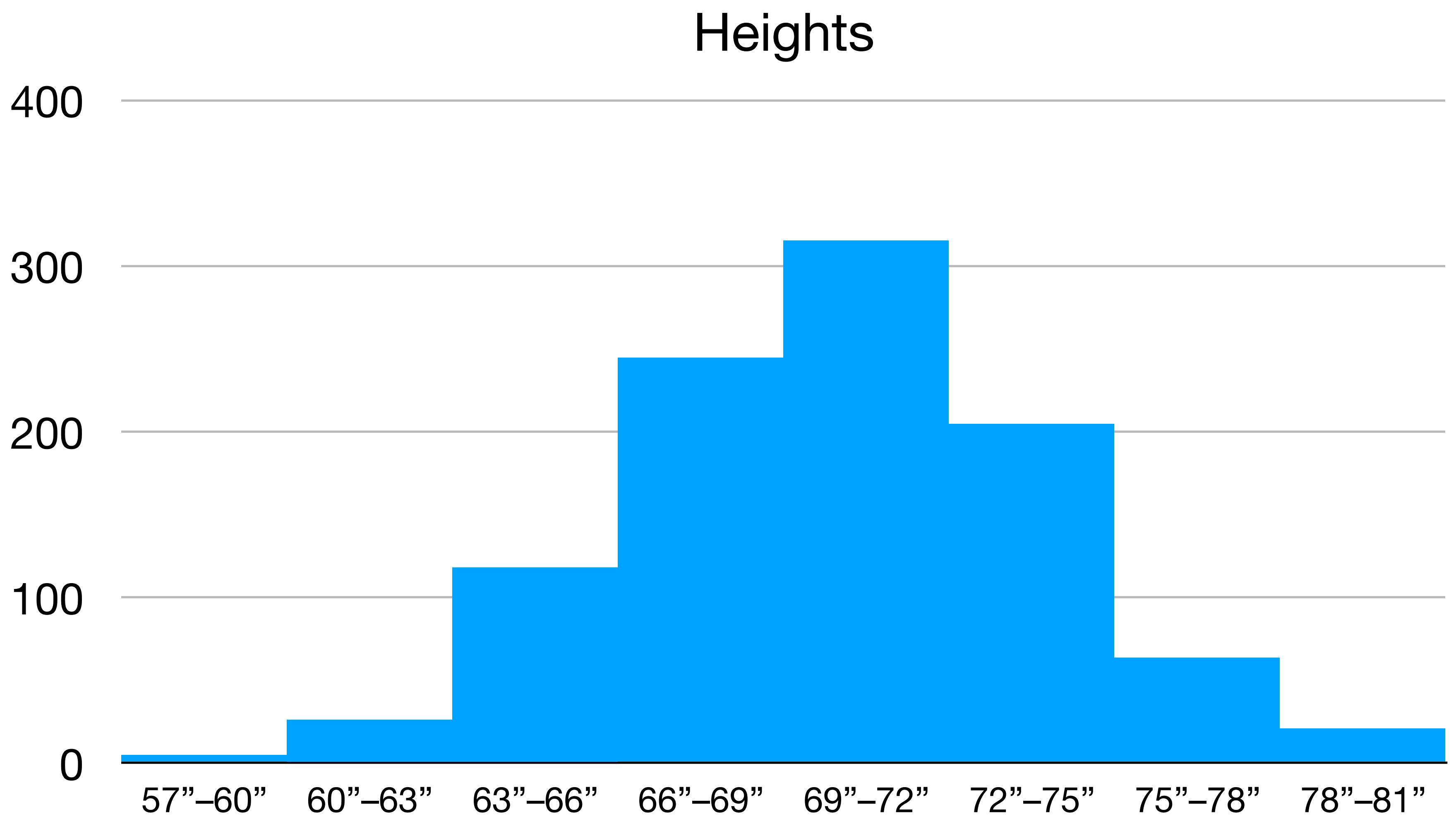
Population μ = 69.436

$$\text{MSE} = \frac{1}{N}\sum_i (\bar{x}_i - \mu)^2$$

MSE of estimates: .118

how good is our estimate?

- What about with smaller samples, e.g., $n = 10$?
- Some $\bar{x}$'s: [68.6, 67.3, 68.7, 68.9, 69.0, 71.5, 69.8, 67.4, 70.0, 70.8]
- Still pretty good estimates, but not quite as good



Population μ = 69.436

$$\text{MSE} = \frac{1}{N}\sum_i (\bar{x}_i - \mu)^2$$

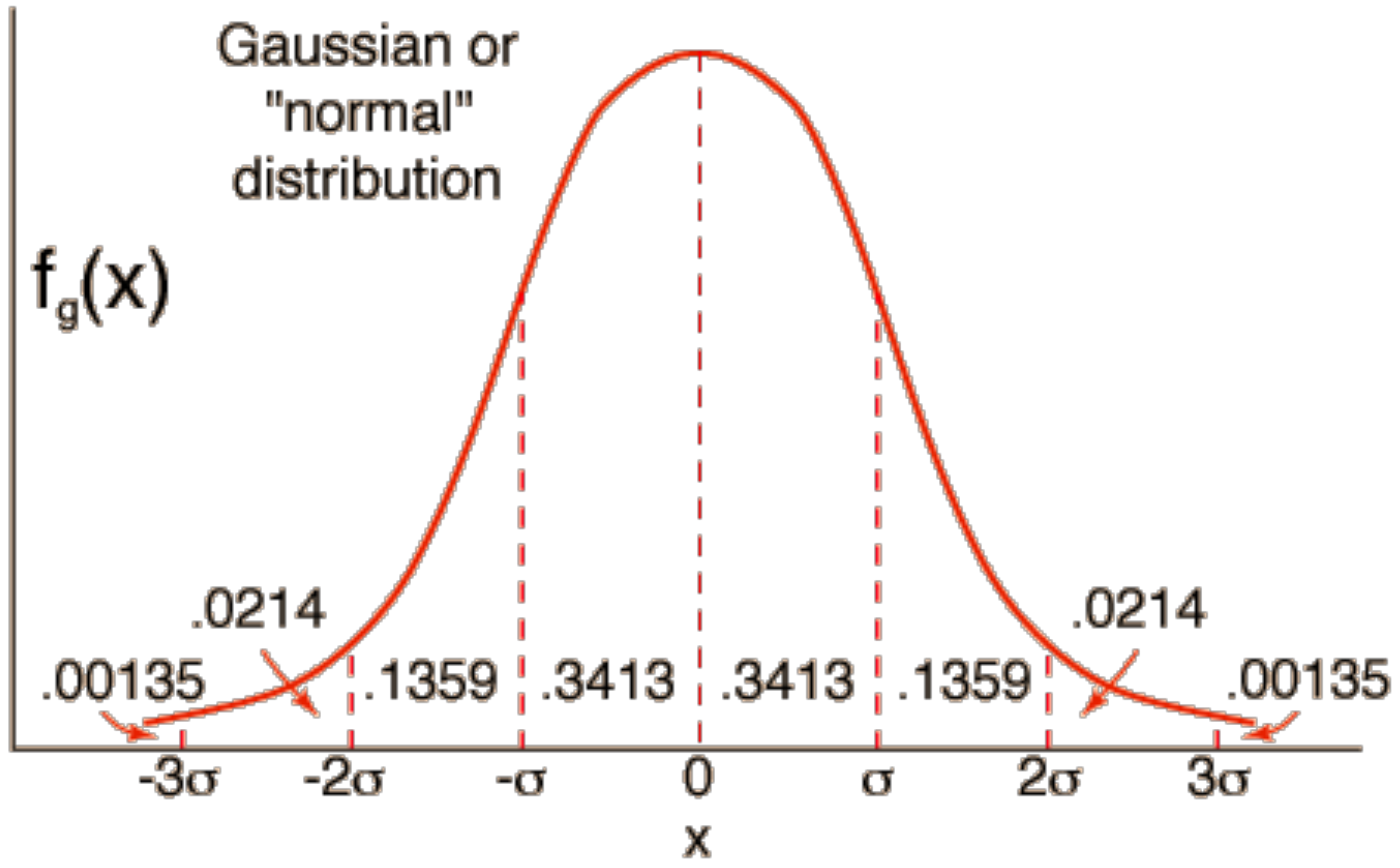
MSE of estimates: 1.70

other useful statistics

- Sample variance (s^2) and standard deviation (s):

$$s^2 = \frac{1}{N-1}\sum_i (x_i - \bar{x})^2, \qquad s = \sqrt{s^2}$$

  - Quantifies the dispersion of the dataset around the mean



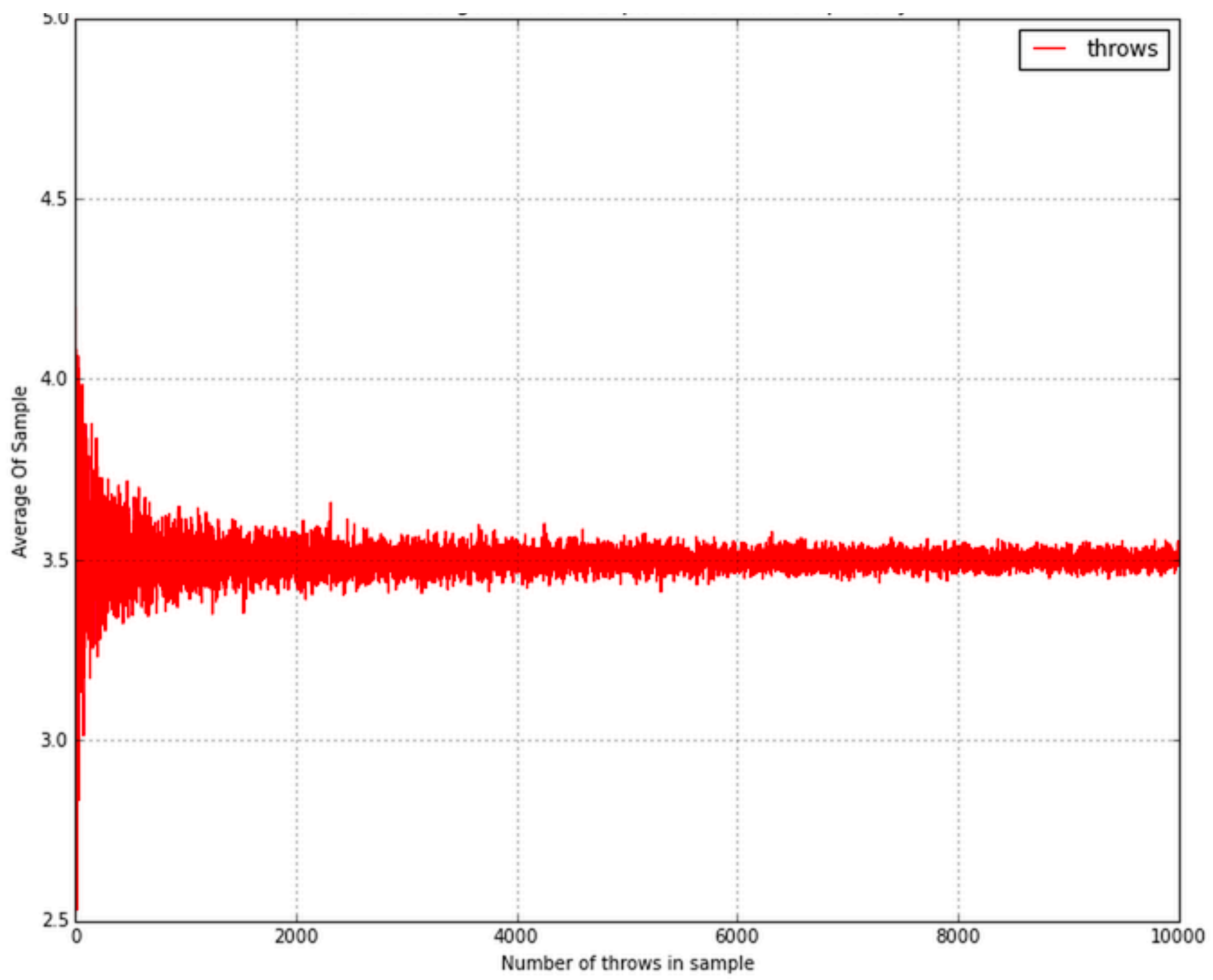
- Why divide by $N-1$ instead of N?
  - Consider the case where $N = 1$ (i.e., one sample), what would be the estimate of s^2?
  - Only $N-1$ degrees of freedom when we are using $\bar{x}$ as the estimate of μ
  - For large N this does not matter much though
- Typically, s^2 is a better estimate of σ^2 than s is of σ. There are several tricks to improve the estimates, but we'll usually just use s directly.

the law of large numbers

- Empirically, we have observed that $\bar{x}$ can be a good estimator for μ
- What we are observing is the **law of large numbers**
    - If $X_1, X_2, ..., X_n$ are independent and identically distributed (**iid**) random variables, then

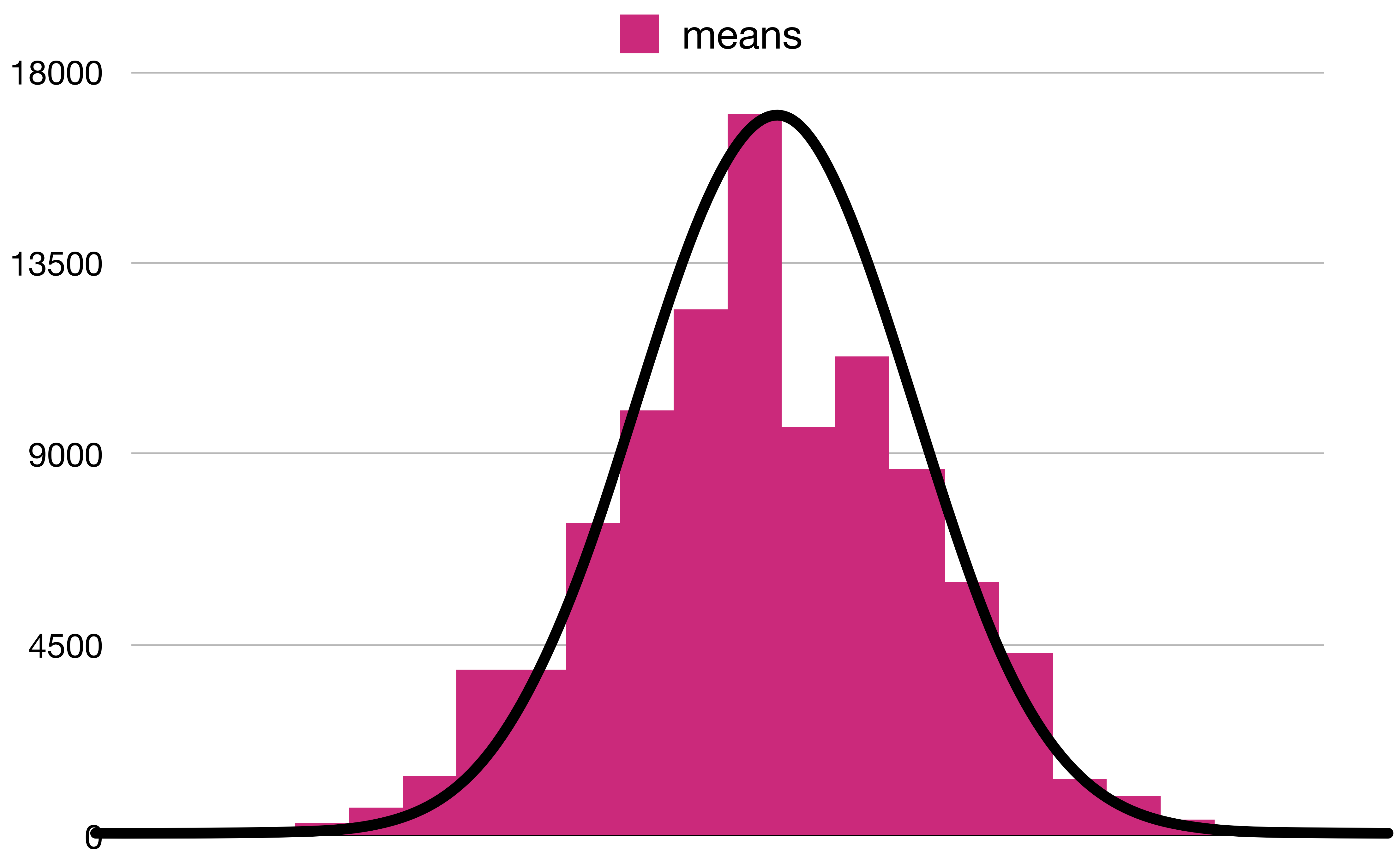
$$\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mu \text{ as } n \to \infty$$

    - In other words, the average of a large number of samples should be close to the population mean
    - But any single sample X_i may still be a bad estimate
- What can I say about how good my estimate is?



sampling distribution

- We can also look at the distribution of a sample statistic, e.g., the mean $\bar{x}$
- This is called a **sampling distribution**
  - View the statistic itself as a random variable
  - Take samples of this variable by running experiments
- Sampling distribution of the sample mean shown on the right
  - It appears to be normally distributed!

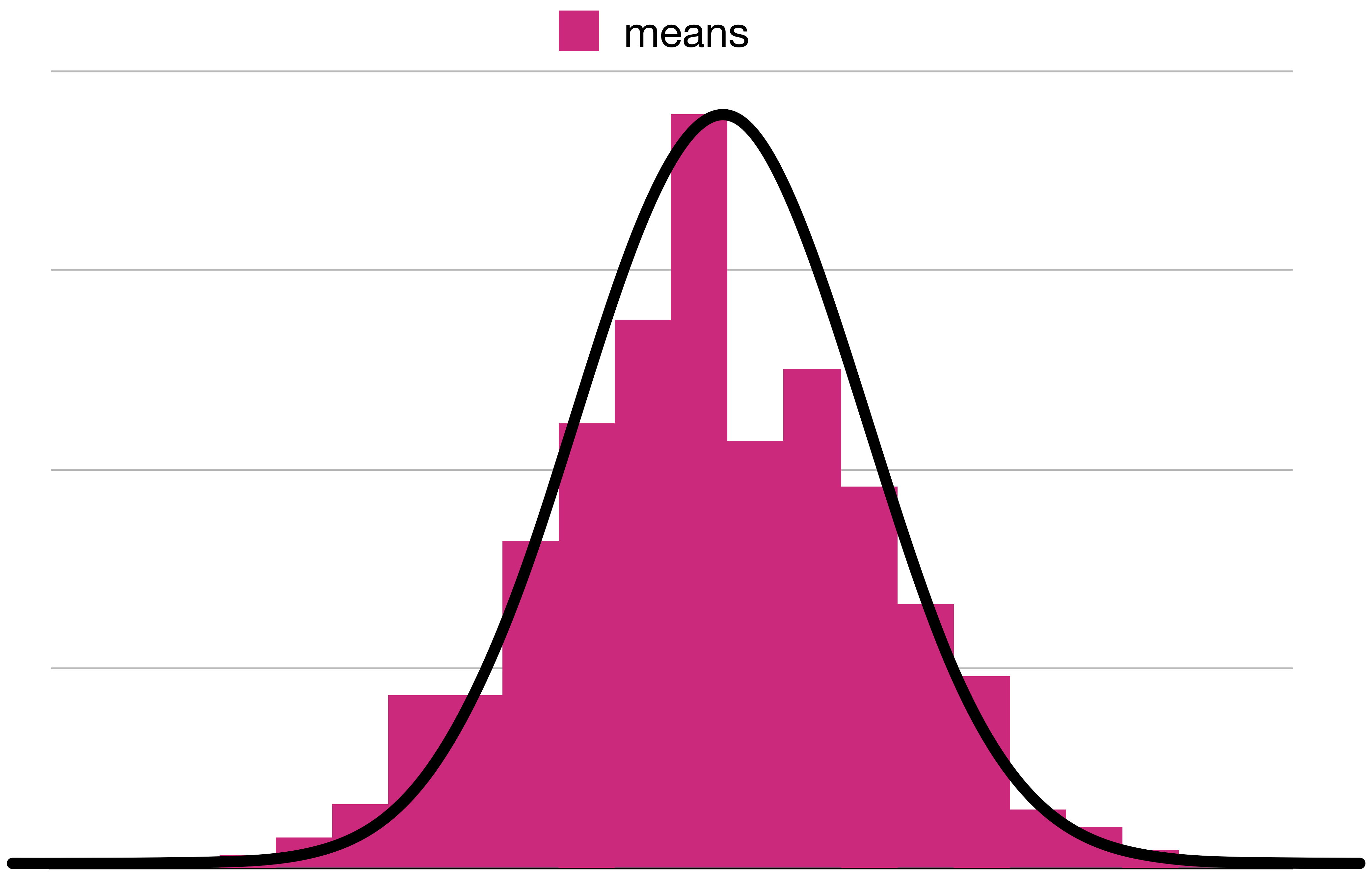


Each data point is the $\bar{x}$ of one experiment

- Average of $\bar{x}$'s = 69.437
- Standard deviation of $\bar{x}$'s = 1.17

central limit theorem

- The sampling distribution of the sample mean is approximately normal
- This is crystalized as the **central limit theorem** (**CLT**)
  - If $X_1, X_2, \ldots, X_n$ are iid random variables, then $\bar{X}_n \rightarrow \mathcal{N}(\mu, \sigma^2/n)$
  - If I take multiple samples from the same distribution, the means tend toward a normal distribution centered on the population mean
- Note: $X_1, X_2, \ldots, X_n$ could have ***any*** distribution (they do **not** need to be normally distributed!)

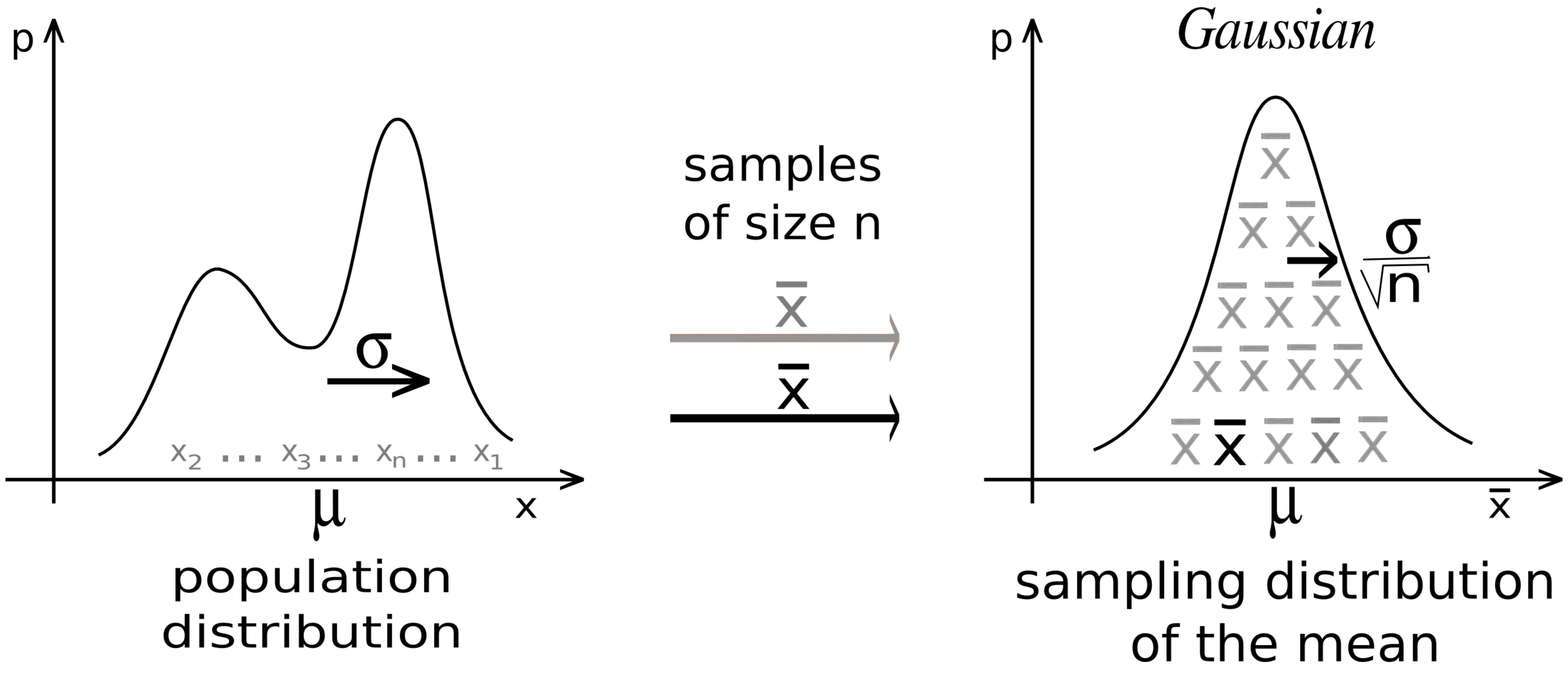


in the limit

- Let's reason directly about the sampling distribution, as if we could repeat the experiment an infinite number of times

- Mean of sampling distribution: μ (the mean of the population)

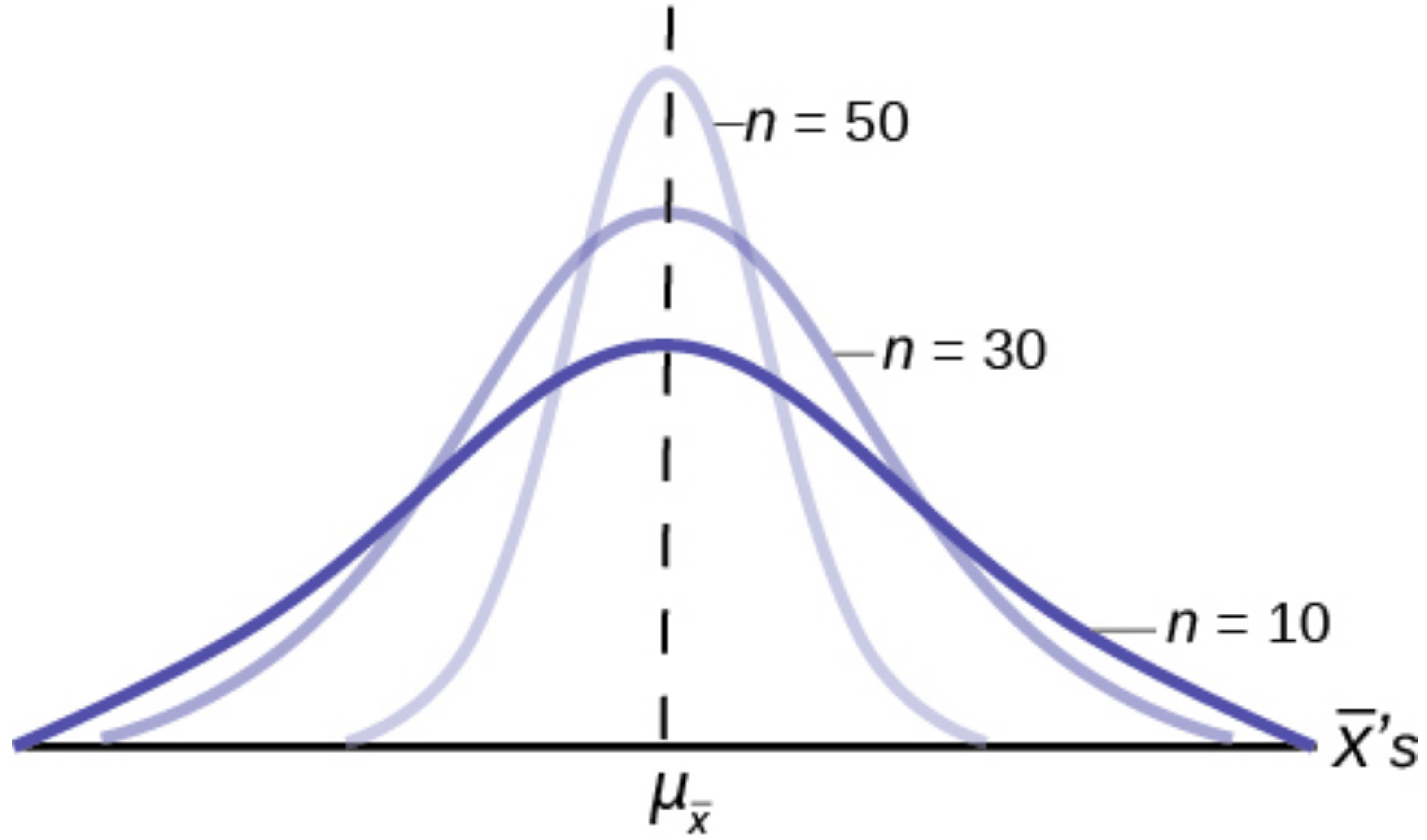
- Variance of sampling distribution: σ^2/n (population variance decaying with n)



- We can approximate the population variance σ^2 by the sample variance s^2 when the size of samples n is large

how does this help us?

- Variance of sampling distribution: σ^2/n
    - The bigger the n (the bigger the samples used to generate the means), the smaller the variance of the sampling distribution (the more tightly clustered the means are)
    - In other words, the bigger your sample, the closer your sample mean is likely to be to the true mean
- Implication: if we have a sample mean (or means), we can use properties of the sampling distribution to let us judge …
    - how good the estimates are (**confidence intervals**)
    - how likely a sample is to be an outlier (**hypothesis testing**)
- Usually we want $n \geq 30$ to say that the CLT holds



example

Suppose that the number of YouTube videos Bob watches each day follows a Binomial distribution with 50 trials and a success probability 0.2.

What is the distribution of the mean number of videos watched among a random sample of 100 days in Bob's life (assuming the days are independent)?

Note that if $X \sim \text{Bin}(k, p)$, then $\mu_X = kp$ and $\sigma_X^2 = kp(1-p)$.

solution

The number of videos Bob watches in a single day follows $X \sim \text{Bin}(50{,}0.2)$. Thus, $\mu_X = 50 \cdot 0.2 = 10$ and $\sigma_X^2 = 50 \cdot 0.2 \cdot 0.8 = 8$.

But we are not interested in X, we are interested in the sampling distribution $\bar{X}_n$ over 100 samples. By the CLT, we know

$$\bar{X}_{100} \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) = \mathcal{N}\left(10, \frac{8}{100}\right) = \mathcal{N}\left(10,\, 0.08\right)$$

Even though X is Binomial, $\bar{X}$ is Gaussian (note that we have a sufficiently large number of samples).